DEGENERATE COMPLEX MONGE-AMPÈRE FLOWS ON STRICTLY PSEUDOCONVEX DOMAINS

DO HOANG SON

ABSTRACT. We study the equation $\dot{u} = \log\det(u_{\alpha\bar{\beta}}) + f(t, z, u)$ in domains of $\mathbb{C}^n$. This equation has a close connection with the Kähler-Ricci flow. In this paper, we consider the case of the boundary conditions are smooth and the initial conditions are bounded.

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Introduction

On Kähler manifolds, a Kähler-Ricci flow is an equation

$$\frac{\partial}{\partial t}\omega = -Ric(\omega), \tag{1}$$

which starts from a Kähler metric. Here, $Ric(\omega)$ is the form associated to the Ricci curvature of ω, i.e., if

$$\omega = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}} dz^i \wedge d\overline{z^j},$$

then

$$Ric(\omega) = -\frac{\sqrt{-1}}{2\pi}(\partial_i \partial_{\bar{j}} \log \det g) dz^i \wedge d\bar{z^j}.$$

This flow was become a poweful tool of geometry. The theory of Kähler-Ricci flow is well developed in the case of compact Kähler manifolds, see e.g. [Cao85], [PS05], [ST07], [Zha09], [Tos10], [GZ13], [BG13]. It can be seen as the parabolic problem associated to an "elliptic" problem which would be the complex Monge-Ampère equation.

Monge-Ampère equations and their generalizations have long been studied in strictly pseudoconvex domains of $\mathbb{C}^n$, see for instance [CKNS85]. This raises a natural question: what is the behavior of the corresponding parabolic equation in the case of $\mathbb{C}^n$?

Let Ω be a bounded smooth strictly pseudoconvex domain of $\mathbb{C}^n$, i.e., there exists a smooth strictly plurisubharmonic function ρ defined on a bounded neighbourhood of $\bar{\Omega}$ such that $\Omega = \{\rho < 0\}$ and $d\rho|_{\partial\Omega} \neq 0$.

Let $T \in (0, \infty]$. We consider the equation

$$\begin{cases} \dot{u} = \log\det(u_{\alpha\bar{\beta}}) + f(t, z, u) & \text{on } \Omega \times (0, T), \\ u = \varphi & \text{on } \partial\Omega \times [0, T), \\ u = u_0 & \text{on } \bar{\Omega} \times \{0\}, \end{cases} \tag{2}$$

where $\dot{u} = \frac{\partial u}{\partial t}$, $u_{\alpha\bar{\beta}} = \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_\beta}$, u_0 is a plurisubharmonic function in a neighbourhood of Ω and f is smooth in $[0, T) \times \bar{\Omega} \times \mathbb{R}$ and non increasing in the last variable.

This equation has a close connection with the Kähler-Ricci flow. There are some previous results. If u_0 is continuous and φ does not depend on the last variable, then (2) admits a unique viscosity solution [EGZ14]. If u_0 is a smooth strictly plurisubharmonic function in $\bar{\Omega}$, φ is smooth in $\bar{\Omega} \times [0, T)$ and the compatibility conditions are satisfied, then (2) admits a unique solution $u \in C^\infty(\Omega \times (0, T)) \cap C^{2;1}(\bar{\Omega} \times [0, T))$ [HL10]; we state their result in detail as Theorem 2.2 in Section 2.

In this paper, we study the case where φ is smooth and u_0 is merely bounded. The main result is the following:

Theorem 0.1. *Let Ω be a bounded smooth strictly pseudoconvex domain of $\mathbb{C}^n$ and $T \in (0, \infty]$. Let u_0 be a bounded plurisubharmonic function defined on a neighbourhood $\tilde{\Omega}$ of $\overline{\Omega}$. Assume that $\varphi \in C^\infty(\bar{\Omega} \times [0, T))$ and $f \in C^\infty([0, T) \times \bar{\Omega} \times \mathbb{R})$ satisfying*

(i) $f_u \leq 0$.

(ii) $\varphi(z, 0) = u_0(z)$ *for* $z \in \partial\Omega$.

Then there exists a unique function $u \in C^\infty(\bar{\Omega} \times (0, T))$ such that

$$u(., t) \text{ is a strictly plurisubharmonic function on } \Omega \quad \text{for all } t \in (0, T), \tag{3}$$

$$\dot{u} = \log\det(u_{\alpha\bar{\beta}}) + f(t,z,u) \ \textit{on}\ \Omega \times (0,T), \tag{4}$$

$$u = \varphi \ \textit{on}\ \partial\Omega \times (0,T), \tag{5}$$

$$\lim_{t\to 0} u(z,t) = u_0(z) \ \ \forall z \in \bar{\Omega}. \tag{6}$$

Moreover, $u \in L^\infty(\bar{\Omega} \times [0,T'))$ *for any* $0 < T' < T$*, and* $u(.,t)$ *also converges to* u_0 *in capacity when* $t \to 0$.
If $u_0 \in C(\tilde{\Omega})$ *then* $u \in C(\bar{\Omega} \times [0,T))$.

Here, we say that $u(.,t)$ converges to u_0 in capacity if the convergence is uniform outside sets of arbitrarily small capacity.

This improves the main result of [HL10] in two directions: we do not need smoothness of the initial data, and still have continuity when $t \to 0$; and we obtain the maximal possible regularity when z tends to $\partial\Omega$, for fixed $t > 0$.

Some techniques used in this paper are from the corresponding result in the case of compact Kähler manifolds. On a compact Kähler manifold, results have been obtained in the more general case where u_0 has zero or even positive Lelong numbers. We refer the reader to [GZ13] and [DL14] for the details.

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1. Strategy of the proof

We fix some notation. We say that $u \in C^{2;1}(\bar{\Omega} \times [0,T))$ if $u(.,t) \in C^2(\bar{\Omega})$ for any $t \in [0,T)$, $u(z,.) \in C^1([0,T))$ for any $z \in \bar{\Omega}$ and $\dot{u}, u_{s_j s_k} \in C(\bar{\Omega} \times [0,T))$ for $s_j, s_k \in \{x_1, y_1, ...x_n, y_n\}$.

In order to prove Theorem 0.1, we use an approximation process and we first will need to prove the following a priori estimates theorem:

Theorem 1.1. *Let* Ω *be a bounded smooth strictly pseudoconvex domain of* $\mathbb{C}^n$ *and* $T > 0$. *Let* $\varphi \in C^\infty(\bar{\Omega} \times [0,T))$ *and* $f \in C^\infty([0,T) \times \bar{\Omega} \times \mathbb{R})$ *and let* $u \in C^\infty(\Omega \times (0,T)) \cap C^{2;1}(\bar{\Omega} \times [0,T))$*, strictly plurisubharmonic with respect to z, be a solution of the equation*

$$\dot{u} = \log\det(u_{\alpha\bar{\beta}}) + f(t,z,u) \quad \textit{on}\ \ \Omega \times (0,T). \tag{7}$$

Assume that

$$u|_{\partial\Omega\times[0,T)} = \varphi|_{\partial\Omega\times[0,T)}, \tag{8}$$

$$\sup |u(z,0)| \le C_u, \tag{9}$$

$$f_u(t,z,u) \le 0 \quad \forall (t,z,u) \in (0,T) \times \Omega \times \mathbb{R}, \tag{10}$$

$$\|f\|_{C^2((0,T)\times\Omega\times\mathbb{R})} \le C_f, \tag{11}$$

$$\|\varphi\|_{C^4(\Omega\times(0,T))} \le C_\varphi. \tag{12}$$

Then there exists $M_0 = M_0(\Omega, T, C_u, C_\varphi, C_f)$ *and for any* $0 < \epsilon < T$ *there exists* $C = C(\Omega, \epsilon, T, C_u, C_\varphi, C_f)$ *such that*

$$|u| \leq M_0 \quad on \quad \Omega \times (0, T),$$
$$|\nabla u| + |\dot{u}| + \Delta u \leq C \quad on \quad \Omega \times (\epsilon, T).$$

Remark 1.2. *In the theorem above, we denote*

$$\|\varphi\|_{C^k(\Omega\times(0,T))} = \sum_{|j|+2l\leq k} \sup_{\Omega\times(0,T)} |D_s^j D_t^l \varphi|,$$

$$\|f\|_{C^k((0,T)\times\Omega\times\mathbb{R}))} = \sum_{j_1+|j_2|+j_3\leq k} \sup |D_t^{j_1} D_s^{j_2} D_u^{j_3} f|,$$

where $s = (s_1, ..., s_{2n}) = (x_1, y_1, ..., x_n, y_n)$.

For the proof of Theorem 0.1, the strategy is as follows.

+ Construct the solutions $u_m \in C^\infty(\Omega \times (0,T)) \cap C^{2;1}(\bar{\Omega} \times [0,T))$ of (4) such that $u_m|_{\bar{\Omega}\times\{0\}}$ and $u_m|_{\partial\Omega\times(0,T)}$ converge pointwise, respectively, to u_0 and $\varphi|_{\partial\Omega\times(0,T)}$. We also ask that the u_m be uniformly bounded and $u_m|_{\partial\Omega\times(\epsilon_m,T)} = \varphi|_{\partial\Omega\times(\epsilon_m,T)}$ for some $\epsilon_m \searrow 0$.
+ Use the a priori estimates to prove
$$\|u_m\|_{C^2(\bar{\Omega}\times(\epsilon,T'))} \leq C_{\epsilon,T'}$$
for any $0 < \epsilon < T' < T$, where $C_{\epsilon,T'} > 0$ is independent of m.
+ Use $C^{2,\alpha}$ estimates and to prove
$$\|u_m\|_{C^k(\bar{\Omega}\times(\epsilon,T'))} \leq C_{k,\epsilon,T'}$$
for any $0 < \epsilon < T' < T$ and $k > 0$, where $C_{k,\epsilon,T'} > 0$ is independent on m. The $C^{2,\alpha}$ estimates and the $C^{k,\alpha}$ regularity will be mentioned in section 5.
+ By Ascoli's theorem, there exists a subsequence of $\{u_m\}$, denoted also by $\{u_m\}$, and $u \in C^\infty(\bar{\Omega} \times (0,T)$ such that
$$u_m \overset{C^k(\bar{\Omega}\times(\epsilon,T'))}{\longrightarrow} u.$$
Then, u satisfies (3), (4) and (5).
+ Use Comparison principle to prove (6).
+ Finally, we prove the uniqueness of u.

We will study some important tools before we prove Theorem 0.1. In Section 2, we introduce some basic results about parabolic complex Monge-Ampère equations. In Sections 3 and 4, we prove the a priori estimates theorem (Theorem 1.1). In Section 5 we establish the $C^{2,\alpha}$ estimate needed to solve our problem. Finally in Section 6 we prove Theorem 0.1.

2. Preliminaries

2.1. **Hou-Li theorem.**

The Hou-Li theorem states that equation (2) has a unique solution when the conditions are good enough. We will use it in Section 6 to obtain smooth solutions to an approximating problem, to which we then will apply the a priori estimates from Theorem 1.1.

We first need the notion of subsolution.

Definition 2.1. *A function* $\underline{u} \in C^\infty(\bar{\Omega} \times [0, T))$ *is called a* subsolution *of the equation* (14) *if and only if*

$$\begin{cases} \underline{u}(., t) \textit{is a strictly plurisubharmonic function,} \\ \dot{\underline{u}} \leq \log\det(\underline{u})_{\alpha\bar{\beta}} + f(t, z, \underline{u}), \\ \underline{u}|_{\partial\Omega\times(0,T)} = \varphi|_{\partial\Omega\times(0,T)}, \\ \underline{u}(., 0) \leq u_0. \end{cases} \tag{13}$$

Theorem 2.2. *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded domain with smooth boundary. Let* $T \in (0, \infty]$. *Assume that*

- φ *is a smooth function in* $\bar{\Omega} \times [0, T)$.
- f *is a smooth function in* $[0, T) \times \bar{\Omega} \times \mathbb{R}$ *non increasing in the lastest variable.*
- u_0 *is a smooth strictly plurisubharmonic funtion in a neighborhood of* Ω.
- $u_0(z) = \varphi(z, 0),\ \forall z \in \partial\Omega$.
- *The compatibility condition is satisfied, i.e.*

$$\dot{\varphi} = \log\det(u_0)_{\alpha\bar{\beta}} + f(t, z, u_0),\ \ \forall (z, t) \in \partial\Omega \times \{0\}.$$

- *There exists a subsolution to the equation* (14).

Then there exists a unique solution $u \in C^\infty(\Omega \times (0, T)) \cap C^{2;1}(\bar{\Omega} \times [0, T))$ *of the equation*

$$\begin{cases} \dot{u} = \log\det(u_{\alpha\bar{\beta}}) + f(t, z, u) & \textit{on } \Omega \times (0, T), \\ u = \varphi & \textit{on } \partial\Omega \times [0, T), \\ u = u_0 & \textit{on } \bar{\Omega} \times \{0\}. \end{cases} \tag{14}$$

Remark 2.3. (i) *There is a corresponding result in the case of a compact Kähler manifold. On the compact Kähler manifold* X, *we must assume that* $0 < T < T_{max}$, *where* T_{max} *depends on* X. *In the case of domain* $\Omega \subset \mathbb{C}^n$, *we can assume that* $T = +\infty$ *if* $\varphi, \underline{u}$ *are defined on* $\bar{\Omega} \times [0, +\infty)$ *and* f *is defined on* $[0, +\infty) \times \bar{\Omega} \times \mathbb{R}$.

(ii) *If* Ω *is a bounded smooth strictly pseudoconvex domain of* $\mathbb{C}^n$ *then one can prove that a subsolution always exists, and so Theorem 2.2 does not need the additional assumpation of existence of a subsolution.*

2.2. **Maximum principle.**

The following maximum principle is a basic tool to establish upper and lower bounds in the sequel (see [BG13] and [IS13] for the proof).

Theorem 2.4. *Let* Ω *be a bounded domain of* C^n *and* $T > 0$. *Let* $\{\omega_t\}_{0<t<T}$ *be a continuous family of continuous positive definite Hermitian forms on* Ω. *Denote by* Δ_t *the Laplacian with respect to* ω_t:

$$\Delta_t f = \frac{n\omega_t^{n-1} \wedge dd^c f}{\omega_t^n},\ \forall f \in C^\infty(\Omega).$$

Suppose that $H \in C^\infty(\Omega \times (0, T)) \cap C(\bar{\Omega} \times [0, T))$ *and satisfies*

$$(\frac{\partial}{\partial t}-\Delta_t)H\leq 0 \ \ or \ \ \dot{H}_t\leq \log\frac{(\omega_t+dd^cH_t)^n}{\omega_t^n}.$$

Then $\sup\limits_{\bar{\Omega}\times[0,T)} H=\sup\limits_{\partial_P(\Omega\times[0,T))} H$. *Here we denote* $\partial_P(\Omega\times(0,T))=\partial\Omega\times(0,T)\cup\bar{\Omega}\times\{0\}$.

Corollary 2.5. *(Comparison principle) Let* Ω *be a bounded domain of* $\mathbb{C}^n$ *and* $T\in(0,\infty]$. *Let* $u,v\in C^\infty(\Omega\times(0,T))\cap C(\bar{\Omega}\times[0,T))$ *satisfying*

- $u(.,t)$ *and* $v(.,t)$ *are strictly plurisubharmonic functions for any* $t\in[0,T)$,
- $\dot{u}\leq\log\det(u_{\alpha\bar{\beta}})+f(t,z,u)$,
- $\dot{v}\geq\log\det(v_{\alpha\bar{\beta}})+f(t,z,v)$,

where $f\in C^\infty([0,T)\times\bar{\Omega}\times\mathbb{R})$ *is non increasing in the last variable.*
Then $\sup\limits_{\Omega\times(0,T)}(u-v)\leq max\{0,\sup\limits_{\partial_P(\Omega\times(0,T))}(u-v)\}$.

Corollary 2.6. *Let* Ω *be a bounded domain of* $\mathbb{C}^n$ *and* $T\in(0,\infty]$. *We denote by* L *a operator on* $C^\infty(\Omega\times(0,T))$ *satisfying*

$$L(f)=\frac{\partial f}{\partial t}-\sum a_{\alpha\bar{\beta}}\frac{\partial^2 f}{\partial z_\alpha\partial\bar{z}_\beta}-b.f,$$

where $a_{\alpha\bar{\beta}},b\in C(\Omega\times(0,T))$, $(a_{\alpha\bar{\beta}}(z,t))$ *are positive definite Hermitian matrices and* $b(z,t)<0$.
Assume that $\phi\in C^\infty(\Omega\times(0,T))\cap C(\bar{\Omega}\times[0,T))$ *satisfies*

$$L(\phi)\leq 0.$$

Then $\phi\leq max(0,\sup\limits_{\partial_P(\Omega\times(0,T))}\phi)$.

2.3. **The Laplacian inequalities.**
We shall need two standard auxiliary results (see [Yau78], [Siu87] for a proof).

Theorem 2.7. *Let* ω_1,ω_2 *be positive* $(1,1)$*-forms on a complex manifold* X*.Then*

$$n\left(\frac{\omega_1^n}{\omega_2^n}\right)^{1/n}\leq tr_{\omega_2}(\omega_1)\leq n\left(\frac{\omega_1^n}{\omega_2^n}\right)(tr_{\omega_1}(\omega_2))^{n-1},$$

where $tr_{\omega_1}(\omega_2)=\dfrac{n\omega_1^{n-1}\wedge\omega_2}{\omega_1^n}$.

Theorem 2.8. *Let* ω, ω' *be two Kähler forms on a complex manifold* X. *If the holomorphic bisectional curvature of* ω *is bounded below by a constant* $B\in\mathbb{R}$ *on* X*,then*

$$\Delta_{\omega'}\log tr_\omega(\omega')\geq-\frac{tr_\omega Ric(\omega')}{tr_\omega(\omega')}+B\,tr_{\omega'}(\omega),$$

where $Ric(\omega')$ *is the form associated to the Ricci curvature of* ω'.

Remark 2.9. *Applying Theorem 2.8 for* $\omega=dd^c|z|^2$ *and* $\omega'=dd^cu$, *we have*

$$\sum u^{\alpha\bar{\beta}}(\log\Delta u)_{\alpha\bar{\beta}}\geq\frac{\Delta\log\det(u_{\alpha\bar{\beta}})}{\Delta u}.$$

2.4. **Construction of subsolutions.**

We give a first construction which will be used in the proof of Theorem 1.1. First we need a notion of subsolution weaker than the one in Definition 2.1.

Definition 2.10. *We say that a function* $\underline{u} \in C^\infty(\bar{\Omega} \times [0,T))$ *is a* subsolution *of the equation* (7) *if*

$$\dot{\underline{u}} \leq \log\det(\underline{u}_{\alpha\bar{\beta}}) + f(t,z,\underline{u}).$$

We will construct subsolutions of (7) in order to prove some estimates on the boundary.

Let $\rho \in SPSH(\bar{\Omega}) \cap C^\infty(\bar{\Omega})$ be a function which defines Ω. We also assume that $\inf \rho = -1$. Let $\zeta \in C^\infty(\mathbb{R})$ such that $0 \leq \zeta \leq 1$, $\zeta|_{[0,1]} = 1$ and $\zeta|_{[2,\infty)} = 0$.

Let φ and u_0 be as in Theorem 1.1. For any $m > 0$, we denote the function $\varphi_m \in C^\infty(\bar{\Omega} \times [0,T))$ by the formula

$$\varphi_m = \varphi - Osc(u_0) \cdot \zeta(mt).$$

Then there exists $M_m > 0$ depending on $\rho, T, C_u, C_\varphi, C_f$ such that the function $\underline{u}_m = \varphi_m + M_m\rho$ satisfies

$$\dot{\underline{u}}_m \leq \log\det(\underline{u}_m)_{\alpha\bar{\beta}} + f(t,z,\underline{u}_m) \text{ on } \Omega \times (0,T),$$
$$dd^c(\underline{u}_m) \geq dd^c|z|^2 \text{ on } \Omega \times [0,T).$$

Then $\underline{u}_m$ is a subsolution of (7). Moreover,

$$\underline{u}_m|_{\partial_P(\Omega\times(0,T))} \leq u|_{\partial_P(\Omega\times(0,T))},$$
$$\underline{u}_m|_{\partial\Omega\times(\frac{2}{m},T)} = \varphi|_{\partial\Omega\times(\frac{2}{m},T)}.$$

By the maximum principle, we have

$$\underline{u}_m \leq u \text{ on } \Omega \times (0,T).$$

In the next two sections, we will prove Theorem 1.1. For convenience, we define an operator L on $C^\infty(\Omega \times (0,T))$ by the formula

$$L(\phi) = \dot{\phi} - \sum u^{\alpha\bar{\beta}}\phi_{\alpha\bar{\beta}} - f_u(t,z,u)\phi, \tag{15}$$

where u is the function in Theorem 1.1 and $(u^{\alpha\bar{\beta}})$ is the transpose of inverse matrix of Hessian matrix $(u_{\alpha\bar{\beta}})$.

3. Order 1 a priori estimates

In this section, we will estimate u, $\dot{u}$ and $|\nabla u|$.

Clearly,

$$\underline{u}_1 \leq u \leq \sup_{\partial\Omega\times(0,T)} \varphi \text{ on } \Omega \times (0,T).$$

Then

$$-M_1 - 2\sup|\varphi| - C_u \leq u(z,t) \leq \sup_{\partial\Omega\times(0,T)} \varphi,$$

where M_1 is the constant defined in 2.4. Let $C_1 = M_1 + 2C_\varphi + C_u$, we obtain

$$\sup|u| \leq C_1. \tag{16}$$

3.1. **Bounds on u̇.**

Proposition 3.1. *There exists $C_2 > 0$ depending only on T, C_f, C_1 such that*

$$t|\dot{u}| \leq C_2 \text{ on } \Omega \times (0, T).$$

Proof. Take L as in (15), then

$$L(t\dot{u} - u) = t\ddot{u} - t\sum u^{\alpha\bar{\beta}}\dot{u}_{\alpha\bar{\beta}} + n - (t\dot{u} - u)f_u(t, z, u).$$

By equation (7), we have

$$t\ddot{u} = t\sum u^{\alpha\bar{\beta}}\dot{u}_{\alpha\bar{\beta}} + t.f_t(t, z, u) + t\dot{u}.f_u(t, z, u).$$

Then

$$-C_2' \leq L(t\dot{u} - u) = n + t.f_t(t, z, u) + u.f_u(t, z, u) \leq C_2',$$

where $C_2' = n + C_f(T + C_1) > 0$.
Since $L(t\dot{u} - u - C_2't) \leq 0$ and $L(t\dot{u} - u + C_2't) \geq 0$, by the maximum principle, we obtain

$$t\dot{u} - u - C_2't \leq \sup_{\partial_P(\Omega\times(0,T))}(t\dot{u} - u - C_2't) \leq (C_\varphi + C_2')T + C_1,$$
$$t\dot{u} - u + C_2't \geq \inf_{\partial_P(\Omega\times(0,T))}(t\dot{u} - u + C_2't) \geq -(C_\varphi + C_2')T - C_1.$$

Thus $t|\dot{u}| \leq C_2$ on $\Omega \times (0, T)$, where $C_2 = (C_\varphi + 2C_2')T + 2C_1$. □

3.2. **Gradient estimates.**

Proposition 3.2. *Let $m > \frac{2}{T}$. Then there exists $C_3 = C_3(\Omega, M_m, C_\varphi) > 0$ such that*

$$|\nabla u| \leq C_3 \text{ on } \partial\Omega \times (\tfrac{2}{m}, T).$$

Proof. Let $h \in C^\infty(\bar{\Omega} \times [0, T))$ be a spatial harmonic function (i.e. harmonic with respect to z) satisfying

$$h = \varphi \text{ on } \partial\Omega \times [0, T).$$

Then taking $\underline{u}_m$ as 2.4 , we have

$$\underline{u}_m \leq u \leq h \text{ on } \Omega \times (\tfrac{2}{m}, T),$$
$$\underline{u}_m = u = h = \varphi \text{ on } \partial\Omega \times (\tfrac{2}{m}, T).$$

Hence

$$|\nabla(u - \underline{u}_m)| \leq |\nabla(h - \underline{u}_m)| \text{ on } \partial\Omega \times (\tfrac{2}{m}, T).$$

Thus

$$|\nabla u| \leq |\nabla\underline{u}_m| + |\nabla(h - \underline{u}_m)| \leq C_3 \text{ on } \partial\Omega \times (\tfrac{2}{m}, T),$$

where $C_3 > 0$ depends only on Ω, C_φ, M_m.

□

Proposition 3.3. *Assume that m, C_3 satisfy Proposition 3.2 and $\frac{2}{m} < \epsilon < T$. Then there exists $C_4 = C_4(\Omega, m, \epsilon, T, C_f, C_1, C_2, C_3) > 0$ such that*

$$|\nabla u| \leq C_4 \text{ on } \Omega \times (\epsilon, T).$$

Proof. We will use the technique of Blocki as in [Blo08]. In this proof only, we denote

$$g(t) = n \log(t - \frac{2}{m}),$$
$$\gamma(u) = Au - Bu^2 \quad \text{where} \quad A = \frac{1}{10C_1}, B = \frac{1}{20C_1^2},$$
$$\eta = \frac{1}{4(\text{diam}\Omega)^2},$$
$$\phi = \log |\nabla u|^2 + \gamma(u) + g(t) + \eta |z|^2,$$

and we assume that $0 \in \Omega$.

Let $\epsilon < T' < T$, we will prove that

$$\sup_{\Omega \times (\frac{2}{m}, T')} \phi \leq \tilde{C}_4,$$

where $\tilde{C}_4$ depends on $\Omega, C_1, C_2, C_3, m, T, C_f$.

Notice that the hypotheses and previous bounds on $|u|$ imply that, for $t \in (\frac{2}{m}, T')$,

$$\exp \phi(z,t) \leq |\nabla u(z,t)|^2 (t - \frac{2}{m})^n \exp \left(\max_{\Omega \times (\frac{2}{m}, T')} \gamma(u) + \eta \max_{\Omega} |z| \right) \leq C |\nabla u|^2, \tag{17}$$

and in a similar way

$$|\nabla u(z,t)|^2 \leq C(\epsilon - \frac{2}{m})^{-n} \exp \phi(z,t) \leq C_\epsilon \exp \phi(z,t), \quad t \in (\epsilon, T'),$$

so the bound on ϕ yields a bound on $|\nabla u(z,t)|$.

Suppose that

$$\sup_{\Omega \times (\frac{2}{m}, T')} \phi = \phi(z_0, t_0).$$

By an orthogonal change of coordinates, we can assume that $(u_{\alpha\bar{\beta}}(z_0, t_0))$ is diagonal. For convenience, we denote $u_{\alpha\bar{\alpha}}(z_0, t_0) = \lambda_\alpha$.

We also denote by $\mathcal{L}$ the operator

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum u^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta}.$$

If $|\nabla u|^2(z_0, t_0) \leq C$, by (17), we are done. In particular, if $z_0 \in \partial\Omega$, we know that $|\nabla u(z,t)|$ is bounded. So we may restrict attention to the case where $|\nabla u|^2(z_0, t_0) > 1$ and $(z_0, t_0) \in \Omega \times (\dfrac{2}{m}, T']$. Then $\mathcal{L}(\phi)|_{(z_0,t_0)} \geq 0$.

We compute

$$\begin{aligned} \mathcal{L}(\phi) &= \mathcal{L}(\log |\nabla u|^2) + \gamma'(u).\dot{u} + g'(t) - \gamma'(u) \sum u^{\alpha\bar{\beta}} u_{\alpha\bar{\beta}} \\ &\quad - \gamma''(u) \sum u^{\alpha\bar{\beta}} u_\alpha u_{\bar{\beta}} - \eta \sum u^{\alpha\bar{\alpha}} \\ &= \mathcal{L}(\log |\nabla u|^2) + \gamma'(u).(\dot{u} - n) + g'(t) \\ &\quad - \gamma''(u) \sum u^{\alpha\bar{\beta}} u_\alpha u_{\bar{\beta}} - \eta \sum u^{\alpha\bar{\alpha}}. \end{aligned}$$

When $|\nabla u|\neq 0$, we have

$$\begin{aligned}
(\log|\nabla u|^2)_{\alpha\bar\beta} &= \frac{|\nabla u|^2_{\alpha\bar\beta}}{|\nabla u|^2}-\frac{|\nabla u|^2_{\alpha}|\nabla u|^2_{\bar\beta}}{|\nabla u|^4}\\
&=\frac{\langle\nabla u_{\alpha\bar\beta},\nabla u\rangle}{|\nabla u|^2}+\frac{\langle\nabla u,\nabla u_{\beta\bar\alpha}\rangle}{|\nabla u|^2}+\frac{\langle\nabla u_{\alpha},\nabla u_{\beta}\rangle}{|\nabla u|^2}\\
&+\frac{\langle\nabla u_{\bar\beta},\nabla u_{\bar\alpha}\rangle}{|\nabla u|^2}-\frac{|\nabla u|^2_{\alpha}|\nabla u|^2_{\bar\beta}}{|\nabla u|^4}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}(\log|\nabla u|^2) &= \frac{\langle\nabla\dot u,\nabla u\rangle-\sum\langle u^{\alpha\bar\beta}\nabla u_{\alpha\bar\beta},\nabla u\rangle}{|\nabla u|^2}+\frac{\langle\nabla u,\nabla\dot u\rangle-\sum\langle\nabla u,u^{\beta\bar\alpha}\nabla u_{\beta\bar\alpha}\rangle}{|\nabla u|^2}\\
&-\sum u^{\alpha\bar\beta}\frac{\langle\nabla u_{\alpha},\nabla u_{\beta}\rangle+\langle\nabla u_{\bar\beta},\nabla u_{\bar\alpha}\rangle}{|\nabla u|^2}+\sum u^{\alpha\bar\beta}\frac{(|\nabla u|^2)_{\alpha}(|\nabla u|^2)_{\bar\beta}}{|\nabla u|^4}.
\end{aligned}$$

We have, by (7),

$$\begin{aligned}
\mathcal{L}(\log|\nabla u|^2)|_{(z_0,t_0)} &= 2Re\left(\frac{\langle\nabla u,\nabla f\rangle}{|\nabla u|^2}\right)+2f_u(t,z,u)|\nabla u|^2-\sum\frac{|\nabla u_k|^2+|\nabla u_{\bar k}|^2}{\lambda_k|\nabla u|^2}\\
&+\sum\frac{(|\nabla u|^2)_k(|\nabla u|^2)_{\bar k}}{\lambda_k|\nabla u|^4}\\
&\leq\frac{2|\nabla f|}{|\nabla u|}+\sum\frac{(|\nabla u|^2)_k(|\nabla u|^2)_{\bar k}}{\lambda_k|\nabla u|^4}.
\end{aligned}$$

Hence, there exists $C_4^{'}=C_4^{'}(m,C_1,C_2,C_f)$ such that

$$\mathcal{L}(\phi)|_{(z_0,t_0)}\leq C_4^{'}+g'(t)-\gamma''(u)\sum\frac{|u_k|^2}{\lambda_k}-\eta\sum\frac{1}{\lambda_k}+\sum\frac{(|\nabla u|^2)_k(|\nabla u|^2)_{\bar k}}{\lambda_k|\nabla u|^4}.$$

By the condition $\frac{\partial\phi}{\partial z_k}(z_0,t_0)=0$, we have

$$\frac{(|\nabla u|^2)_k(|\nabla u|^2)_{\bar k}}{|\nabla u|^4}=|\gamma'(u)u_k+\eta\bar z_k|^2\leq 2(\gamma'(u))^2|u_k|^2+2\eta^2|z_k|^2\leq 2(\gamma'(u))^2|u_k|^2+\frac{\eta}{2},$$

where $(z,t)=(z_0,t_0)$.
Then

$$\begin{aligned}
0\leq\mathcal{L}(\phi)|_{(z_0,t_0)} &\leq C_4^{'}+g'(t)+(2(\gamma'(u))^2-\gamma''(u))\sum\frac{|u_k|^2}{\lambda_k}-\frac{\eta}{2}\sum\frac{1}{\lambda_k}\\
&\leq C_4^{'}+g'(t)-a(\sum\frac{|u_k|^2}{\lambda_k}+\sum\frac{1}{\lambda_k}),
\end{aligned}$$

where $a:=\min\{2B-(A+BC_1),\frac{\eta}{2}\}$. Hence, at (z_0,t_0)

$$\sum\frac{|u_k|^2}{\lambda_k}+\sum\frac{1}{\lambda_k}\leq\frac{1}{a}(C_4^{'}+g'(t)). \tag{18}$$

Moreover, by Proposition 3.1 and by (16), there exists $C_4^{''}=C_4^{''}(m,C_1,C_2)$ such that

$$\lambda_1\lambda_2...\lambda_n=\det(u_{\alpha\bar\beta})=e^{\dot u-f(t,z,u)}\leq C_4^{''}. \tag{19}$$

By (18) and (19), there exists $C_4^{'''} = C_4^{'''}(a, C_4^{'}, C_4^{''})$ such that

$$\lambda_k = \prod \lambda_j \prod_{l\neq k} \frac{1}{\lambda_l} \leq (C_4^{'''} + g'(t_0))^{n-1} \quad \text{for} \quad k = 1, ..., n.$$

$$|\nabla u|^2 = \sum |u_k|^2 \leq ((C_4^{'''} + g'(t_0))^n \quad \text{for} \quad (z,t) = (z_0, t_0).$$

Then

$$\begin{aligned} \phi(z_0, t_0) &\leq n \log(C_4^{'''} + g'(t_0)) + g(t_0) + \gamma(u(z_0, t_0)) + \eta|z_0|^2 \\ &\leq n \log(C_4^{'''}(t_0 - \tfrac{2}{m}) + n) + \gamma(u(z_0, t_0)) + \eta|z_0|^2 \\ &\leq \tilde{C}_4. \end{aligned}$$

For $z \in \Omega$, $\frac{2}{m} < \epsilon < t < T'$, we have

$$\log |\nabla u|^2 \leq \tilde{C}_4 - \gamma(u) - \eta|z|^2 - g(t) \leq 2 \log C_4,$$

where $C_4 > 0$ depends on $\Omega, m, \epsilon, T, C_f, C_1, C_2, C_3$. □

4. Higher order estimates

In this section, we prove that the second derivatives of u are bounded on $\partial\Omega \times (\epsilon, T)$. Then we use the maximum principle to show that the Laplacian of u is bounded on $\Omega \times (\epsilon, T)$. For convenience, we denote $\underline{u} := \underline{u}_m$, $M := M_m$, where $\frac{1}{2m} < \epsilon \leq \frac{1}{2m-1}$ and u_m, M_m are defined as in 2.4.

4.1. **Localisation technique.**

In order to show that the second derivatives of u are bounded on $\partial\Omega \times (\epsilon, T)$, we use a barrier function. The key to the construction is the following:

Lemma 4.1. *We set*

$$v = (u - \underline{u}) + a(h - \underline{u}) - Nd^2,$$

where d is the distance from $\partial\Omega$, h is defined as in the proof of Proposition 3.2 and a, N are positive constants to be determined. Let $\epsilon \in (0, T)$. Then there exist $a, N, \delta > 0$ depending only on $\Omega, \epsilon, T, C_u, C_\varphi, C_f$ such that

$$L(v) \geq \tfrac{1}{4}(1 + \textstyle\sum u^{\alpha\bar{\alpha}}) \quad \text{on} \quad U_\delta \times (\epsilon, T),$$

$$v \geq 0 \quad \text{on} \quad U_\delta \times (\epsilon, T),$$

where $U_\delta = \{z \in \Omega : d(z) \leq \delta\}$.

Proof. The elliptic version of this lemma was proved by [Gua98] (page 5-7). The same arguments can be applied for the parabolic case. For the reader's convenience, we recall the arguments here.
We have

$$L(v) = \dot{v} - n + \sum u^{\alpha\bar{\beta}} \underline{u}_{\alpha\bar{\beta}} - a \sum u^{\alpha\bar{\beta}}(h_{\alpha\bar{\beta}} - \underline{u}_{\alpha\bar{\beta}}) + 2N \sum u^{\alpha\bar{\beta}}(dd_{\alpha\bar{\beta}} + d_\alpha d_{\bar{\beta}}) - f_u(t, z, u)v.$$

Fix $\tilde{\delta} > 0$ satisfying $d \in C^\infty(U_{\tilde{\delta}})$. Assume that $0 < a < 1$ and $0 < \delta < \tilde{\delta}$ and $0 < N < \frac{1}{\delta}$. Then there exists $C_5 > 0$ depending on $\Omega, \tilde{\delta}, \epsilon, T, C_\varphi, C_f, M, C_1, C_2$ such that

$$\begin{gathered}\dot{v} - n - f_u(t,z,u)v \geq -C_5,\\ -a\sum u^{\alpha\bar{\beta}}(h_{\alpha\bar{\beta}} - \underline{u}_{\alpha\bar{\beta}}) \geq -C_5 a\sum u^{\alpha\bar{\alpha}},\\ 2Nd\sum u^{\alpha\bar{\beta}} d_{\alpha\bar{\beta}} \geq -C_5 N\delta\sum u^{\alpha\bar{\alpha}},\end{gathered}$$

where $(z,t) \in U_\delta \times (\epsilon, T)$.

Then

$$L(v) \geq \sum u^{\alpha\bar{\beta}}\underline{u}_{\alpha\bar{\beta}} - C_5 - C_5(a+N\delta)\sum u^{\alpha\bar{\alpha}} + 2N\sum u^{\alpha\bar{\beta}} d_\alpha d_{\bar{\beta}},$$

where $(z,t) \in U_\delta \times (\epsilon, T)$.

When $a + N\delta \leq \frac{1}{4C_5}$, we obtain

$$L(v) \geq \frac{3}{4}\sum u^{\alpha\bar{\alpha}} - C_5 + 2N\sum u^{\alpha\bar{\beta}} d_\alpha d_{\bar{\beta}},$$

where $(z,t) \in U_\delta \times (\epsilon, T)$.

Let $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be the eigenvalues of $\{u_{\alpha\bar{\beta}}\}$. We have

$$\sum u^{\alpha\bar{\beta}} d_\alpha d_{\bar{\beta}} \geq \lambda_n^{-1}\sum d_\alpha d_{\bar{\alpha}} \geq \frac{\lambda_n^{-1}}{2} \quad \text{on} \quad U_\delta \times (\epsilon, T).$$

By the inequality for arithmetic and geometric means

$$\frac{1}{4}\sum u^{\alpha\bar{\alpha}} + N\lambda_n^{-1} \geq n(\frac{1}{4})^{(n-1)/n} N^{1/n}(\lambda_1 ... \lambda_n)^{-1/n} \geq C_6 N^{1/n},$$

where $C_6 > 0$ depends on $\epsilon, T, C_f, C_1, C_2$.
When $N > (\frac{C_5+1}{C_6})^n$, we have

$$L(v) \geq \frac{1}{2}(2 + \sum u^{\alpha\bar{\alpha}}).$$

Next, since $\Delta\underline{u} \geq n$, there exists $C_7 > 0$ depending only on Ω such that

$$(h - \underline{u}) \geq C_7 d \quad \text{on} \quad \Omega \times (\epsilon, T).$$

Fix $0 < a, \delta < 1$, $N > 0$ so that

- $N > \left(\dfrac{C_5+1}{C_6}\right)^n$;
- $a \leq \dfrac{1}{8C_5}$;
- $0 < \delta < \tilde{\delta}$;
- $\min\{aC_7, a\} \geq N\delta$.

We obtain

$$L(v) \geq \tfrac{1}{4}(1 + \sum u^{\alpha\bar{\alpha}}) \quad \text{on} \quad U_\delta \times (\epsilon, T),$$

$$v \geq 0 \quad \text{on} \quad U_\delta \times (\epsilon, T).$$

□

4.2. $\mathcal{C}^2$**-a priori estimates on the boundary.**

Lemma 4.2. *Let $\epsilon \in (0,T)$. Then there exists $c_\epsilon > 0$ depending only on $\Omega, \epsilon, T, C_u, C_\varphi, C_f$ such that*

$$(dd^c u)|_{T^h_{\partial\Omega}} \geq c_\epsilon (dd^c|z|^2)|_{T^h_{\partial\Omega}},$$

where $T^h_{\partial\Omega}$ is the holomorphic tangent bundle of $\partial\Omega$.

We refer the reader to [CKNS85, pp. 221–223] or [Bou11, p. 268–271] for related results in the elliptic case.

Proof. Fix $p \in \partial\Omega$. By an affine change of coordinates, we can assume that $p = 0$ and there exists a neighbourhood U of p such that

$$\Omega \cap U = \{z \in U : x_n > Re(\sum_{1\leq j\leq k\leq n} a_{j\bar{k}} z_j \bar{z}_k + \sum_{1\leq j\leq k\leq n} a_{jk} z_j z_k) + O(|z|^3)\},$$

where $a_{j\bar{k}}, a_{jk} \in \mathbb{C}$ with $a_{1\bar{1}} > 0$.

By a holomorphic change of coordinates, we can assume that

$$\Omega \cap U = \{z \in U : x_n > Re(\sum_{1\leq j\leq k\leq n} a_{j\bar{k}} z_j \bar{z}_k) + O(|z|^3)\}, \tag{20}$$

where $a_{j\bar{k}}$ with $a_{1\bar{1}} > 0$.

We need to show that

$$u_{1\bar{1}}(p,t) \geq C_\epsilon,$$

where $t \in (\epsilon, T)$ and $C_\epsilon > 0$ depends on $\Omega, \epsilon, T, C_u, C_\varphi, C_f$.

Step 1: Choice of a Kähler potential.

We construct a function $\tau \in C^\infty(\Omega_r \times (\epsilon, T))$ depending on $\underline{u}, \epsilon, T, \Omega$ so that $dd^c\tau = dd^c\underline{u}$ and $\tau(p,t) = 0$ and

$$\tau|_{(\partial\Omega\cap B_r)\times(\epsilon,T)} = Re\left(\sum_{j=2}^n c_j z_1 \bar{z}_j\right) + O\left(|z_2|^2 + ... + |z_n|^2\right),$$

where $r > 0$, $B_r = B_r(p)$, $\Omega_r = \Omega \cap B_r$ and $c_j \in C^\infty([\epsilon, T), \mathbb{C})$.

Indeed, by Taylor's formula,

$$\begin{aligned}\underline{u}(z,t) - \underline{u}(p,t) &= Re(\sum_{j=1}^n b_j z_j) + Re(\sum_{j=2}^n b_{1\bar{j}} z_1 \bar{z}_j) + b_{1\bar{1}}|z_1|^2 + Re(\sum_{j=1}^n b_{1j} z_1 z_j) \\ &\quad + O(|z_2|^2 + ... + |z_n|^2) + O(|z|^3),\end{aligned}$$

where $b_j, b_{1j}, b_{1\bar{j}} \in \mathbb{C}^\infty([\epsilon, T), \mathbb{C})$, $b_{1\bar{1}} = \underline{u}_{1\bar{1}}(p,t) > 0$.

Furthermore, near p on $\partial\Omega$, we have by (20)

$$x_n = Re(\sum_{j=2}^n a_{1\bar{j}} z_1 \bar{z}_j) + a_{1\bar{1}}|z_1|^2 + O(|z_2|^2 + ... + |z_n|^2) + O(|z|^3), \tag{21}$$

where $a_{1\bar{j}} \in \mathbb{C}$ with $a_{1\bar{1}} > 0$.

Define

$$\tau(z,t) = \underline{u}(z,t) - \underline{u}(p,t) - Re(\sum_{j=1}^n b_j z_j) - \frac{b_{1\bar{1}}}{a_{1\bar{1}}} x_n - Re(\sum_{j=1}^n b_{1j} z_1 z_j);$$

then $dd^c\tau = dd^c\underline{u}$ and $\tau(p,t)=0$ and

$$\tau|_{(\partial\Omega\cap B_r)\times(\epsilon,T)} = Re\left(\sum_{j=2}^n c_j z_1\bar{z}_j\right) + O(|z_2|^2+...+|z_n|^2) + \{\text{terms of order} \geq 3\}.$$

Moreover, for $z\in\partial\Omega$, we have

- For $j=2,...,n$

$$|z_j|^2|z_1| = O(|z_2|^2+...+|z_n|^2); \tag{22}$$

- By (21)

$$\begin{aligned}|z_1|^4 &= O(x_n^2) + O(\sum_{j=2}^n |z_1|^2|z_j|^2) + O(|z|^6) + O((\sum_{j=2}^n |z_j|^2)^2)\\ &= O(|z_2|^2+...+|z_n|^2) + O(|z|^6);\end{aligned}$$

  then

$$|z|^4 = O(|z_2|^2+...+|z_n|^2); \tag{23}$$

- For $j=2,...,n$

$$|z_1|^2|z_j| = O(|z_1|^4) + O(|z_j|^2) = O(|z_2|^2+...+|z_n|^2). \tag{24}$$

Hence

$$\begin{aligned}\tau|_{(\partial\Omega\cap B_r)\times(\epsilon,T)} &= Re(\sum_{j=2}^n c_j z_1\bar{z}_j) + \sum \tilde{a}_j x_1^j y_1^{3-j} + O(|z_2|^2+...+|z_n|^2)\\ &= Re(\sum_{j=2}^n c_j z_1\bar{z}_j) + Re(a_1 z_1^3) + Re(a_2 z_1|z_1|^2) + O(|z_2|^2+...+|z_n|^2),\end{aligned}$$

where $a_1, a_2 \in C^\infty([\epsilon,T),\mathbb{C})$.
Next, by (21), (22), (24), for $z\in\partial\Omega$, we have

$$\begin{aligned}Re(a_2 z_1|z_1|^2) &= Re(\frac{a_2}{a_{1\bar{1}}} z_1 x_n) + O(|z_2|^2+...+|z_n|^2)\\ &= Re(c_0 z_1\bar{z}_n) + Re(c_0 z_1 z_n) + O(|z_2|^2+...+|z_n|^2).\end{aligned}$$

Replacing the term c_n by $c_n - c_0$, we obtain

$$\tau|_{(\partial\Omega\cap B_r)\times(\epsilon,T)} = Re\left(\sum_{j=2}^n c_j z_1\bar{z}_j\right) + Re(a_1 z_1^3) + Re(c_0 z_1 z_n) + O(|z_2|^2+...+|z_n|^2).$$

Replacing τ by $\tau + Re(a_1 z_1^3) + Re(c_0 z_1 z_n)$, we obtain

$$\tau|_{(\partial\Omega\cap B_r)\times(\epsilon,T)} = Re\left(\sum_{j=2}^n c_j z_1\bar{z}_j\right) + O(|z_2|^2+...+|z_n|^2).$$

Therefore,

$$\tau|_{(\partial\Omega\cap B_r)\times(\epsilon,T)} \leq Re\left(\sum_{j=2}^n c_j z_1\bar{z}_j\right) + a_3(|z_2|^2+...+|z_n|^2), \quad \sup\sum_{j=2}^n |c_j| \leq a_4, \tag{25}$$

where $a_3, a_4 > 0$ depend on $\Omega, \epsilon, T, M, C_\varphi$.
The conditions $dd^c\tau = dd^c\underline{u}$ and $\tau(p,t)=0$ are still satisfied.

Step 2: Choice of a barrier function.
Recall that $\Omega_r = \Omega \cap B_r$. We construct a function

$$b(z,t) = -\epsilon_1 x_n + \epsilon_2|z|^2 + \frac{1}{2\mu}\sum_{j=2}^{n}|c_j z_1 + \mu z_j|^2 \tag{26}$$

such that $b \geq \tau + u - \underline{u}$ on $\Omega_r \times (\epsilon, T)$, where $r > 0$ depends only on Ω and $\epsilon_1, \epsilon_2, \mu > 0$ depend on $\Omega, \epsilon, T, M, C_\varphi, C_f$.
Note that

$$|z_1|^2 \leq \frac{1}{a_{1\bar{1}}}(x_n - Re(\sum_{j=2}^{n} a_{1\bar{j}} z_1 \bar{z}_j)) + O(|z_2|^2 + ... + |z_n|^2) + O(|z|^3) \text{ on } \Omega.$$

Since for r_0 small enough and $z \in \Omega_{r_0}$,we have $z \to 0$ as $|z_2|^2 + ... + |z_n|^2 \to 0$, if we fix $r > 0$ small enough, then there exists $r_1 > 0$ such that

$$|z_2|^2 + ... + |z_n|^2 \geq r_1 \quad \text{for} \quad z \in \partial B_r \cap \Omega.$$

Assume that $0 < \epsilon_1, \epsilon_2 < 1$. Then there exists $\mu_1 > 0$ depending on $\Omega, M, C_\varphi, C_1, a_3, a_4, r_1$ such that the function b in (26) verifies

$$\begin{aligned}
b|_{(\partial B_r(p)\cap\Omega)\times[\epsilon,T)} &\geq \frac{\mu r_1}{2} + Re(\sum_{j=2}^{n} c_j z_1 \bar{z}_j) - \epsilon_1 x_n + \epsilon_2|z|^2 \\
&\geq \frac{\mu_1 r_1}{2} + Re(\sum_{j=2}^{n} c_j z_1 \bar{z}_j) - \epsilon_1 x_n + \epsilon_2|z|^2 \\
&\geq (\tau + u - \underline{u})|_{(\partial B_r(p)\cap\Omega)\times[\epsilon,T)}
\end{aligned}$$

when $\mu \geq \mu_1$.
There exists $r_2 > 0$ such that, when $z \in \partial\Omega$,

$$x_n = Re(\sum_{j=1}^{n} a_{1\bar{j}} z_1 \bar{z}_j) + O(|z_2|^2 + ... + |z_n|^2) + O(|z|^3) \leq r_2|z|^2.$$

Assume that $0 < r_2\epsilon_1 < \epsilon_2$. For $\mu \geq 2a_3$, by (25), we have

$$\begin{aligned}
b|_{(\partial\Omega\cap B_r(p))\times[\epsilon,T)} &\geq \frac{1}{2\mu}\sum_{j=2}^{n}|c_j z_1 + \mu z_j|^2 \\
&\geq Re(\sum_{j=2}^{n} c_j z_1 \bar{z}_j) + \frac{\mu}{2}(|z_2|^2 + ... + |z_n|^2) \\
&\geq \tau|_{(\partial\Omega\cap B_r(p))\times[\epsilon,T)} \\
&\geq (\tau + u - \underline{u})|_{(\partial\Omega\cap B_r(p))\times[\epsilon,T)}.
\end{aligned}$$

Fix $\mu \geq \max(\mu_1, 2a_3)$, we get

$$b|_{\partial_P(\Omega_r\times[\epsilon,T))} \geq (\tau + u - \underline{u})|_{\partial\Omega_r\times[\epsilon,T)}.$$

Next, by Proposition 3.1 ,there exists $r_3 > 0$ such that

$$(dd^c(\tau - u - \underline{u}))^n = (dd^c u)^n = e^{\dot{u} - f(t,z,u)} \geq r_3 \quad \text{on} \quad \Omega_r \times [\epsilon, T).$$

On the other hand

$$(dd^c(\sum_{j=2}^{n}|c_jz_1+\mu z_j|^2))^n=0,$$

so $(dd^cb)^n=O(\epsilon_2)$ on $\Omega_r\times[\epsilon,T)$.
Hence, there exists $\epsilon_2>0$ depending on μ,Ω,a_4,r_3 such that

$$(dd^cb)^n\leq(dd^c(\tau+u-\underline{u}))^n \text{ on } \Omega_r\times[\epsilon,T).$$

When $b|_{\partial\Omega_r\times[\epsilon,T)}\geq(\tau+u-\underline{u})|_{\partial\Omega_r\times[\epsilon,T)}$ and $(dd^cb)^n\leq(dd^c(\tau+u-\underline{u}))^n$ on $\Omega_r\times[\epsilon,T)$, it follows from the comparison theorem (for the bounded plurisubharmonic functions) that

$$b\geq(\tau+u-\underline{u})\quad\text{on}\quad\Omega_r\times[\epsilon,T).$$

Step 3: Conclusion.
We have, since $b(p,t)=\tau(p,t)+u(p,t)-\underline{u}(p,t)=0$,

$$-\epsilon_1=b_{x_n}(p,t)\geq\tau_{x_n}(p,t)+(u-\underline{u})_{x_n}(p,t).$$

Then, since $(u-\underline{u})|_{\partial\Omega\times(\epsilon,T)}\equiv0$,

$$(u-\underline{u})_{1\bar{1}}(p,t)=-(u-\underline{u})_{x_n}(p,t)\rho_{1\bar{1}}(p),$$

and by the explicit choice of τ, $-\tau_{x_n}(p,t)\rho_{1\bar{1}}(p)=\tau_{1\bar{1}}(p,t)$, so

$$u_{1\bar{1}}(p,t)=(\tau_{1\bar{1}}+u_{1\bar{1}}-\underline{u}_{1\bar{1}})(p,t)=-\left(\tau_{x_n}(p,t)+(u-\underline{u})_{x_n}(p,t)\right)\rho_{1\bar{1}}(p)\geq\epsilon_1\rho_{1\bar{1}}(p).$$

□

Proposition 4.3. *There exists $D_1=D_1(\Omega,\epsilon,T,C_u,C_\varphi,C_f)$ such that*

$$|D^2u|\leq D_1\quad\textit{on}\quad\partial\Omega\times(\epsilon,T).$$

Proof. Fix $p\in\partial\Omega$. We can choose complex coordinates $(z_j)_{1\leq j\leq n}$ so that $p=0$ and the positive x_n axis is the interior normal direction of $\partial\Omega$ at p. We set for convenience

$$s_1=y_1,s_2=x_1,...,s_{2n-1}=y_n,s_{2n}=x_n,s'=(s_1,...,s_{2n-1}).$$

We also assume that near p, $\partial\Omega$ is represented as a graph

$$x_n=P(s')=\sum_{j,k<2n}P_{jk}s_js_k+O(|s'|^3).$$

Step 1: Bounding the tangent-tangent derivatives.
Since $(u-\underline{u})(s',P(s'),t)=0$, we have for $j,k<2n,\ 0<t<T$:

$$(u-\underline{u})_{s_js_k}(p,t)=-(u-\underline{u})_{x_n}(p,t)P_{jk}.$$

By Proposition 3.2, we obtain

$$|u_{s_js_k}(p,t)|\leq D_1',$$

where $D_1'>0$ depends only on Ω,C_φ,M.
Step 2: Bounding the normal-tangent derivatives.

Define

$$T_j = \frac{\partial}{\partial s_j} + P_{s_j}\frac{\partial}{\partial x_n}.$$

Again, denote $\Omega_\delta = B_\delta(p)\cap\Omega$. With v as in Lemma 4.2, we construct the functions

$$\psi_\pm = Av + B|z|^2 - (t-\tfrac{\epsilon}{2})(u_{y_n}-\underline{u}_{y_n})^2 \pm (t-\tfrac{\epsilon}{2})T_j(u-\underline{u}),$$

such that

$$\begin{array}{c} L(\psi_\pm)\ge 0 \quad \text{on} \quad \Omega_\delta\times(\tfrac{\epsilon}{2},T), \\ \psi_\pm \ge 0 \quad \text{on} \quad \Omega_\delta\times(\tfrac{\epsilon}{2},T), \end{array}$$

where $A, B > 0$ depend on $\Omega, C_\varphi, C_f, \epsilon, T, M$.
We compute

$$\begin{aligned} L(-(u_{y_n}-\underline{u}_{y_n})^2) &= -2(u_{y_n}-\underline{u}_{y_n})L(u_{y_n}-\underline{u}_{y_n}) - f_u(t,z,u)(u_{y_n}-\underline{u}_{y_n})^2 \\ &\quad +2\sum u^{\alpha\bar\beta}(u_{y_n}-\underline{u}_{y_n})_\alpha(u_{y_n}-\underline{u}_{y_n})_{\bar\beta} \end{aligned}$$

and

$$\begin{aligned} L(\pm T_j(u-\underline{u})) &= \pm L(u_{s_j}-\underline{u}_{s_j}) \pm P_{s_j}L(u_{x_n}-\underline{u}_{x_n}) \\ &\quad \mp(u_{x_n}-\underline{u}_{x_n})\sum u^{\alpha\bar\beta}(P_{s_j})_{\alpha\bar\beta} \\ &\quad \mp\sum u^{\alpha\bar\beta}\left((u_{x_n}-\underline{u}_{x_n})_\alpha(P_{s_j})_{\bar\beta} + (u_{x_n}-\underline{u}_{x_n})_{\bar\beta}(P_{s_j})_\alpha\right). \end{aligned}$$

By equation (7), for $k = 1, 2, ..., 2n$

$$L(u_{s_k}-\underline{u}_{s_k}) = f_{s_k}(t,z,u) - \dot{\underline{u}}_{s_k} + \sum u^{\alpha\bar\beta}(\underline{u}_{s_k})_{\alpha\bar\beta} + \underline{u}_{s_k}f_u(t,z,u).$$

Hence

$$\begin{aligned} & L(-(u_{y_n}-\underline{u}_{y_n})^2 \pm T_j(u-\underline{u})) \\ & \ge -C_8(1+\sum u^{\alpha\bar\alpha}) + 2\sum u^{\alpha\bar\beta}(u_{y_n}-\underline{u}_{y_n})_\alpha(u_{y_n}-\underline{u}_{y_n})_{\bar\beta} \\ & \mp\sum u^{\alpha\bar\beta}\left((u_{x_n}-\underline{u}_{x_n})_\alpha(P_{s_j})_{\bar\beta} + (u_{x_n}-\underline{u}_{x_n})_{\bar\beta}(P_{s_j})_\alpha\right), \end{aligned}$$

where $C_8 > 0$ depend on $\epsilon, C_1, C_2, C_3, M, C_\varphi, C_f, \rho, P$.
On the other hand

$$\begin{aligned} \sum_{\alpha=1}^n u^{\alpha\bar\beta}u_{x_n\alpha} &= 2\delta_{\beta n} - i\sum_{\alpha=1}^n u^{\alpha\bar\beta}u_{y_n\alpha}, \\ \sum_{\beta=1}^n u^{\alpha\bar\beta}u_{x_n\bar\beta} &= 2\delta_{\alpha n} + i\sum_{\beta=1}^n u^{\alpha\bar\beta}u_{y_n\bar\beta}. \end{aligned}$$

Then

$$\begin{aligned} & L(-(u_{y_n}-\underline{u}_{y_n})^2 \pm T_j(u-\underline{u})) \\ & \ge -C_9(1+\sum u^{\alpha\bar\alpha}) + 2\sum u^{\alpha\bar\beta}(u_{y_n}-\underline{u}_{y_n})_\alpha(u_{y_n}-\underline{u}_{y_n})_{\bar\beta} \\ & \mp\sum u^{\alpha\bar\beta}\left((u_{y_n}-\underline{u}_{y_n})_\alpha(-iP_{s_j})_{\bar\beta} + (u_{y_n}-\underline{u}_{y_n})_{\bar\beta}(iP_{s_j})_\alpha\right) \end{aligned}$$

where $C_9 > 0$ depend on $\epsilon, C_1, C_2, C_3, M, C_\varphi, C_f, \rho, P$.
By the Cauchy-Schwarz inequality,
$2\sum u^{\alpha\bar{\beta}}(u_{y_n} - \underline{u}_{y_n})_\alpha (u_{y_n} - \underline{u}_{y_n})_{\bar{\beta}} + \dfrac{1}{2}\sum u^{\alpha\bar{\beta}}(iP_{s_j})_\alpha(-iP_{s_j})_{\bar{\beta}}$
$\geq \pm \sum u^{\alpha\bar{\beta}}\big((u_{y_n} - \underline{u}_{y_n})_\alpha(-iP_{s_j})_{\bar{\beta}} + (u_{y_n} - \underline{u}_{y_n})_{\bar{\beta}}(iP_{s_j})_\alpha\big)$.
Then

$$L(-(u_{y_n} - \underline{u}_{y_n})^2 \pm T_j(u - \underline{u})) \geq -C_{10}(1 + \sum u^{\alpha\bar{\alpha}}),$$

where $C_{10} > 0$ depends on $\Omega, C_\varphi, C_f, \epsilon, T, M$.
Hence, by Lemma 4.2, we can choose $A, B > 0$ independent of u so that

$$\begin{aligned} L(\psi_\pm) \geq 0 \quad &\text{on} \quad \Omega_\delta \times (\epsilon, T), \\ \psi_\pm \geq 0 \quad &\text{on} \quad \partial_P(\Omega_\delta \times (\epsilon, T)). \end{aligned}$$

By the maximum principle, we obtain $\psi_\pm \geq 0$ on $\Omega_\delta \times (\frac{\epsilon}{2}, T)$.

Note that $\psi_\pm(p, t) = 0$ for $t \in (\frac{\epsilon}{2}, T)$.

Hence,

$$\lim_{x_n \searrow 0} \frac{\psi_\pm(p + (0, \ldots, x_n), t) - \psi_\pm(p, t)}{x_n} \geq 0,$$

thus

$$|u_{s_j x_n}(p, t)| \leq D_1'',$$

where $t \in (\epsilon, T)$ and $D''_1 > 0$ depend only on $\Omega, C_\varphi, C_f, \epsilon, T, C_u$.
Step 3:Bounding the normal-normal derivatives.
We have that

$$\det(u_{\alpha\bar{\beta}}) = e^{\dot{u} - f(t,z,u)}$$

is bounded from above and below on $\partial\Omega \times (\epsilon, T)$.
By step 1 and step 2, $|u_{z_n\bar{z}_n}\det(u_{\alpha\bar{\beta}})_{\alpha,\beta\leq n-1}|$ is bounded on $\{p\} \times (\epsilon, T)$.
Hence, by Lemma 4.2 , we obtain

$$|u_{z_n\bar{z}_n}(p, t)| \leq D_1''' \ , \ t \in (\epsilon, T),$$

where D_1''' depends on $\Omega, C_\varphi, C_f, \epsilon, T, C_u$.
Consequently

$$|u_{x_n x_n}| \leq D_1'''',$$

where D_1'''' depends on $\Omega, C_\varphi, C_f, \epsilon, T, C_u$.

□

4.3. **Interior estimate of the Laplacian.**

Proposition 4.4. *There exists $D_2 = D_2(\Omega, \epsilon, T, C_\varphi, C_f, C_u)$ such that*

$$\Delta u \leq D_2 \quad \textit{on} \quad \Omega \times (\epsilon, T).$$

Proof. We set

$$\phi = (t - \epsilon)\log\Delta u + A_1|z|^2 - A_2 t,$$

where $A_1, A_2 > 0$ will be specified later.
We have

$$\begin{aligned}L(\phi) \quad &= \log \Delta u + (t-\epsilon)\frac{\Delta \dot{u}}{\Delta u} - A_2 - (t-\epsilon)\sum u^{\alpha\bar{\beta}}(\log \Delta u)_{\alpha\bar{\beta}} \\ &-A_1 \sum u^{\alpha\bar{\alpha}} - \phi f_u(t,z,u).\end{aligned}$$

By Theorem 2.7,

$$\log \Delta u \leq \log n + \log\det(u_{\alpha\bar{\beta}}) + (n-1)\log(\sum u^{\alpha\bar{\alpha}}).$$

By Theorem 2.8,

$$\begin{aligned}\frac{\Delta \dot{u}}{\Delta u} - \sum u^{\alpha\bar{\beta}}(\log \Delta u)_{\alpha\bar{\beta}} \quad &\leq \frac{\Delta \dot{u}}{\Delta u} - \frac{\Delta \log\det(u_{\alpha\bar{\beta}})}{\Delta u} \\ &= \frac{\Delta f(t,z,u)}{\Delta u} \\ &= \frac{\Delta_z f(t,z,u)}{\Delta u} + f_u(t,z,u) + \sum \frac{f_{us_j}(t,z,u)u_{s_j}}{\Delta u} \\ &+ \sum \frac{f_{uu}(t,z,u)u_{s_j}^2}{\Delta u}.\end{aligned}$$

Hence, there exist $A_1, A_2 > 0$ depending on $\Omega, \epsilon, T, C_\varphi, C_f, C_u$ such that

$$L(\phi) \leq 0 \text{ on } \Omega \times (\epsilon, T).$$

Thus, by the maximum principle and Proposition 4.3,

$$(t-\epsilon)\log \Delta u \leq D_2' \quad \text{on} \quad \Omega \times (\epsilon, T),$$

where D_2' depends on $\Omega, \epsilon, T, C_\varphi, C_f, C_u$.
Therefore,

$$\Delta u \leq e^{D_2'/\epsilon} \quad \text{on} \quad \Omega \times (2\epsilon, T).$$

□

5. $C^{2,\alpha}$ ESTIMATE UP TO THE BOUNDARY FOR THE PARABOLIC EQUATION

5.1. **Parabolic Hölder spaces.**

The reader can find more complete notations in [Lieb96, Chapter 4] or [Kryl96, Chapter 8].

In $\mathbb{R}^N \times \mathbb{R}$ we define the parabolic distance between the points $X_1 = (x_1, t_1)$, $X_2 = (x_2, t_2)$ as

$$d(X_1, X_2) = |x_1 - x_2| + |t_1 - t_2|^{1/2}.$$

Let $0 < \alpha < 1$. Let u be a function defined in a domain $Q \subset \mathbb{R}^N \times \mathbb{R}$. We say that u is uniformly Hölder continuous in Q with exponent α, or $u \in C^\alpha(Q)$, if and only if

$$[u]_{\alpha;Q} = \sup_{X_j \in Q, X_1 \neq X_2} \frac{|u(X_1) - u(X_2)|}{d^\alpha(X_1, X_2)} < \infty.$$

Let $0 < \beta < 2$. We denote

$$\langle u\rangle_{\beta;Q}=\sup_{(x,t_1)\neq(x,t_2)\in Q}\frac{|u(x,t_1)-u(x,t_2)|}{|t_1-t_2|^{\beta/2}}.$$

We say that u is uniformly Hölder continuous in Q with exponent $k+\alpha$, or $u\in C^{k,\alpha}(Q)$ if the derivatives $D_x^jD_t^lu$ exist for $|j|+2l\leq k$ and the norm

$$\|u\|_{C^{k,\alpha}(Q)}=\sum_{|j|+2l\leq k}\sup_Q|D_x^jD_t^lu|+\sum_{|j|+2l=k}[D_x^jD_t^lu]_{\alpha;Q}+\sum_{|j|+2l=k-1}\langle D_x^jD_t^lu\rangle_{\alpha+1;Q}$$

is finite.
The norm $\|.\|_{C^{k,\alpha}(Q)}$ makes $C^{k,\alpha}(Q)$ a Banach space. If we define the similar notions for $\bar{Q}$, then $C^{k,\alpha}(Q)=C^{k,\alpha}(\bar{Q})$.

5.2. $\mathbf{C^{2,\alpha}}$ **estimate up to the boundary.**
Let Ω be a bounded smooth domain of $\mathbb{R}^N$. We consider the equation

$$\dot{u}=F(D^2u)+f(t,x,u)\text{ in }\Omega\times(0,\tilde{T}),\tag{27}$$

where $\tilde{T}>0$, f is a smooth function defined on $[0,\tilde{T})\times\bar{\Omega}\times\mathbb{R}$ and F is a smooth concave function defined on the set of all real $N\times N$ matrices. In addition, we assume that there exist $0<\lambda<\Lambda<\infty$ such that

$$\lambda\ \mathrm{tr}\eta\leq F(r+\eta)-F(r)\leq\Lambda\ \mathrm{tr}\eta\tag{28}$$

for any symmetric matrix r, any positive definite matrix η.
We will establish $C^{2,\alpha}$ estimates for the solution of (27) on $\bar{\Omega}\times(\epsilon,T)$ for any $0<\epsilon<T<\tilde{T}$ without $C^{2,\alpha}$ conditions on $\Omega\times\{0\}$. The main result of this section is the following:

Theorem 5.1. *Let F be concave and smooth satisfying* (28). *Let f be a smooth function in $[0,\tilde{T})\times\bar{\Omega}\times\mathbb{R}$ and φ be a smooth function in $\bar{\Omega}\times[0,\tilde{T})$. Assume that $u\in C^{2;1}(\bar{\Omega}\times[0,\tilde{T}))\cap C^\infty(\Omega\times(0,\tilde{T}))$ is a solution of*

$$\begin{cases}\dot{u}=F(D^2u)+f(t,x,u) & \text{in }\Omega\times(0,\tilde{T}),\\ u=\varphi & \text{on }\partial\Omega\times(0,\tilde{T}),\end{cases}\tag{29}$$

and that

$$|u|+|\dot{u}|+|\nabla u|+|D^2u|\leq C,$$

then $u\in C^{2,\alpha}(\bar{\Omega}\times(0,\tilde{T}))$ satisfies

$$\|u\|_{C^{2,\alpha}(\Omega\times(\epsilon,T))}\leq C_{\epsilon,T}\quad\forall 0<\epsilon<T<\tilde{T},\tag{30}$$

where $0<\alpha<1$, $C_{\epsilon,T}>0$ depend on $\lambda,\Lambda,\Omega,C,\epsilon,T$ and the upper bound of $\|\varphi\|_{C^4}+\|F\|_{C^1}+\|f\|_{C^2}$.

Remark 5.2. *In the theorem above, we denote*

$$\|\varphi\|_{C^k(\Omega\times(0,\tilde{T}))}=\sum_{|j|+2l\leq k}\sup_{\Omega\times(0,\tilde{T})}|D_x^jD_t^l\varphi|,$$

$$\|F\|_{C^k(Mat(N\times N,\mathbb{R}))}=\sum_{|j|\leq k}\sup|D^jF|,$$

$$\|f\|_{C^k((0,\tilde{T})\times\Omega\times\mathbb{R}))} = \sum_{j_1+|j_2|+j_3\leq k} \sup |D_t^{j_1} D_x^{j_2} D_u^{j_3} f|.$$

In order to prove Theorem 5.1, we use the technique of Caffarelli as in [CC95]. We need to prove a series of lemmas.

Lemma 5.3. *There exist* $0 < \beta < 1$ *and* $C_{\epsilon,T} > 0$ *depending on* $\lambda, \Lambda, \Omega, C, \epsilon, T$ *and the upper bound of* $\|\varphi\|_{C^4} + \|F\|_{C^1} + \|f\|_{C^1}$ *such that*

$$\frac{\|D^2u(x,t) - D^2u(x_0,t_0)\|}{(|x-x_0| + |t-t_0|^{1/2})^\beta} \leq C_{\epsilon,T}, \quad \forall x, x_0 \in \partial\Omega; \forall t, t_0 \in (\epsilon, T).$$

Proof. Let $x_0 \in \partial\Omega$. We consider a smooth diffeomorphism

$$\begin{aligned} \psi : U \cap \Omega &\longrightarrow B_4^+ := \{y \in \mathbb{R}^N : |y| < 4, y_N > 0\} \\ x &\mapsto y = \psi(x) \end{aligned}$$

such that $\psi(x_0) = 0$ and

$$\psi(U \cap \partial\Omega) = \Gamma_4 = \{y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |y'| < 4, y_N = 0\},$$

where U is a neighborhood of x_0.
We define

$$v(y,t) = u(\psi^{-1}(y), t) - \varphi(\psi^{-1}(y), t),$$

where $y \in B_4^+ \bigcup \Gamma_4$, $t \in (\epsilon, T)$. Then $v|_{\Gamma_4\times(\epsilon,T)} = 0$ and v satisfies the equation

$$\dot{v} = G(t, y, v, Dv, D^2v) \tag{31}$$

where the upper bound of $\|G\|_{C^1}$ depends on $\|F\|_{C^1}$, $\|f\|_{C^1}$ and ψ. Moreover,there exists $A > 1$ depending on ψ (hence, A depends only on Ω) such that

$$\frac{\lambda}{A}|\xi|^2 \leq \frac{\partial G}{\partial r_{ij}}\xi_i\xi_j \leq A\Lambda|\xi|^2$$

for all $\xi \in \mathbb{R}^N$.
Now we only need to show

$$\|D^2v(y,t) - D^2v(0,t_0)\| \leq C_{\epsilon,T}(|y| + |t-t_0|^{1/2})^\beta$$

for any $y \in \Gamma_1, t, t_0 \in (\epsilon, T)$.
By the implicit function theorem, we have

$$v_{NN} = H(t, y, v, \dot{v}, Dv, (v_{ij})_{j<N}).$$

By the chain rule, we have

$$|DH| \leq \frac{A}{\lambda}(\sup |DG| + 1).$$

Hence, there exists $B > 0$ such that

$$\begin{aligned} |v_{NN}(y,t) - v_{NN}(0,t_0)| \leq\ & B(\sup_{j<N} |v_{ij}(y,t) - v_{ij}(0,t_0)| + |\dot{v}(y,t) - \dot{v}(0,t_0)| \\ & + |Dv(y,t) - Dv(0,t_0)| + |y| + |t-t_0|). \end{aligned}$$

Note that $\dot{v}|_{\Gamma_4\times(\epsilon,T)} = v_j|_{\Gamma_4\times(\epsilon,T)} = v_{ij}|_{\Gamma_4\times(\epsilon,T)} = 0$ for $j < N$. Then we only need to show

$$|v_N(y,t) - v_N(0,t_0)| \leq C_{\epsilon,T}(|y| + |t-t_0|^{1/2})^\beta, \tag{32}$$

$$|v_{Nk}(y,t) - v_{Nk}(0,t_0)| \leq C_{\epsilon,T}(|y| + |t-t_0|^{1/2})^\beta, \tag{33}$$

for any $y \in \Gamma_1, t, t_0 \in (\epsilon, T)$ and $k < N$.
By (31), we have

$$\dot{v} = \Delta v + f_1(t,y), \tag{34}$$

where Δ is the Laplacian operator and $f_1(t,y) = G(t,y,v,Dv,D^2v) - \Delta v$. By the hypothesis of theorem, $\|f_1\|_{L^\infty}$ is bounded by a universal constant.
Now we take the derivative of equation (31) in the direction y_k and get that

$$\dot{v}_k = \sum_{i,j=1}^N (v_k)_{ij} \frac{\partial G}{\partial r_{ij}}(t,y,v,Dv,D^2v) + f_2(t,y), \tag{35}$$

where

$$f_2(t,y) = \frac{\partial G}{\partial y_k}(t,y,v,Dv,D^2v) + v_k \frac{\partial G}{\partial p}(t,y,v,Dv,D^2v) + \sum_{l=1}^N v_{lk} \frac{\partial G}{\partial q_l}(t,y,v,Dv,D^2v).$$

Then $\|f_2\|_{L^\infty}$ is bounded by a universal constant.
Then [Lieb96, Lemma 7.32] states that

Lemma 5.4. *If $u \in C^{2;1}(B_4^+ \times (0,T))$ satisfies*

$$|\dot{u} - \sum a_{ij} u_{ij}| \leq A_1,$$
$$|u| \leq A_2 x_N,$$

where $a_{ij} \in C(B_4^+ \times (0,T))$ is such that

$$\sup |a_{ij}| \leq B \text{ and}$$
$$\lambda |\xi|^2 \leq \sum a_{ij}\xi_i\xi_j \leq \Lambda |\xi|^2,$$

then there are positive constants β and C determined only by $A_1, A_2, B, \lambda, \Lambda, \epsilon, T, N$ such that

$$(\sup_{U(y,t,R)} \frac{u}{x_N} - \inf_{U(y,R)} \frac{u}{x_N}) \leq CR^\beta \left(\sup_{B_4^+\times(0,T)} \frac{u}{x_N} - \inf_{B_4^+\times(0,T)} \frac{u}{x_N} + 1 \right),$$

where $y \in B_1^+$, $2\epsilon < t < T - 2\epsilon$, $R < \epsilon$ and $U(y,t,R) = B_R^+(y) \times (t - R^2, t + R^2)$.

Applying this lemma to the equations (34) and (35), we obtain (32) and (33). $\square$

Corollary 5.5. *There exists $C_{\epsilon,T} > 0$ depending on $\lambda, \Lambda, \Omega, C, \epsilon, T$ and the upper bound of $\|\varphi\|_{C^4} + \|F\|_{C^1} + \|f\|_{C^1}$ such that*

$$\frac{|\dot{u}(x,t) - \dot{u}(x_0,t_0)|}{(|x-x_0| + |t-t_0|^{1/2})^\beta} \leq C_{\epsilon,T}, \quad \forall x, x_0 \in \partial\Omega; \forall t, t_0 \in (\epsilon, T).$$

where $0 < \beta < 1$ is the constant in Lemma 5.3.

Lemma 5.6. *There exists $C_{\epsilon,T} > 0$ depending on $\lambda, \Lambda, \Omega, C, \epsilon, T$ and the upper bound of $\|\varphi\|_{C^4} + \|F\|_{C^1} + \|f\|_{C^1}$ such that*

$$\frac{|\dot{u}(x,t) - \dot{u}(x_0,t_0)|}{(|x-x_0| + |t-t_0|^{1/2})^{\beta/2}} \le C_{\epsilon,T}, \quad \forall x \in \Omega, x_0 \in \partial\Omega; \forall t, t_0 \in (\epsilon, T).$$

where $0 < \beta < 1$ is the constant in Lemma 5.3.

Proof. By equation (29), we have

$$|\ddot{u} - \sum \frac{\partial F}{\partial r_{ij}} \dot{u}_{ij}| = |f_t(t,x,u) + \dot{u} f_u(t,x,u)| \le A, \tag{36}$$

where $A > 0$ is a universal constant.

Let $x_0 \in \partial\Omega$ and $t_0 \in (2\epsilon, T)$. We can choose coordinates $(x_j)_{1\le j\le N}$ so that $x_0 = 0$ and the positive x_N axis is the interior normal direction of $\partial\Omega$ at x_0. We also assume that near x_0, $\partial\Omega$ is represented as a graph

$$x_N = P(x') = \sum_{j,k<N} P_{jk} x_j x_k + O(|x'|^3),$$

where $x' = (x_1, ..., x_{N-1})$.

Let $Q(x') = P(x') - |x'|^2$. We consider

$$v = K_1(x_N - Q(x'))^{\beta/2} + K_2((x_N - Q(x'))^2 + (t_0 - t))^{\beta/4}.$$

We have

$$\begin{aligned}\frac{\partial^2 (x_N - Q(x'))^{\beta/2}}{\partial x_i \partial x_j} &= \frac{\beta(\beta-2)}{4}(x_N - Q(x'))^{\beta/2-2} \frac{\partial (x_N - Q(x'))}{\partial x_i} \frac{\partial (x_N - Q(x'))}{\partial x_j} \\ &+ \frac{\beta}{2}(x_N - Q(x'))^{\beta/2-1} \frac{\partial^2 (x_N - Q(x'))}{\partial x_i \partial x_j},\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial^2 ((x_N - Q(x'))^2 + t_0 - t)^{\beta/4}}{\partial x_i \partial x_j} \\ &= \frac{\beta(\beta-4)}{4}((x_N - Q(x'))^2 + t_0 - t)^{\beta/4-2}(x_N - Q(x'))^2 \frac{\partial (x_N - Q(x'))}{\partial x_i} \frac{\partial (x_N - Q(x'))}{\partial x_j} \\ &+ \frac{\beta}{4}((x_N - Q(x'))^2 + t_0 - t)^{\beta/4-1} \frac{\partial^2 (x_N - Q(x'))^2}{\partial x_i \partial x_j}.\end{aligned}$$

Hence, there exists $R > 0$ satisfying, by $F_{r_{11}} \ge \lambda$,

$$\sum_{i,j=1}^N \frac{\partial F}{\partial r_{ij}} \frac{\partial^2 (x_N - Q(x'))^{\beta/2}}{\partial x_i \partial x_j} \le \frac{\lambda\beta(\beta-2)}{6}(x_N - Q(x'))^{\beta/2-2} < 0, \tag{37}$$

and

$$\sum_{i,j=1}^N \frac{\partial F}{\partial r_{ij}} \frac{\partial^2 ((x_N - Q(x'))^2 + t_0 - t)^{\beta/4}}{\partial x_i x_j} = O(x_N - Q(x'))^{\beta/2-2}. \tag{38}$$

On the other hand,

$$|\dot{u} - \dot{u}(0,t_0)|\,|_{\partial_P((\Omega\cap B_R)\times(\epsilon,t_0))} = O(((x_N - Q(x'))^2 + t_0 - t)^{\beta/4}). \tag{39}$$

By (36), (37), (38), (39), there exists $K_1, K_2 > 0$ such that

$$v|_{\partial_P((\Omega\cap B_R)\times(\epsilon,t_0))} \geq \pm(\dot{u} - \dot{u}(0,t_0))|_{\partial_P((\Omega\cap B_R)\times(\epsilon,t_0))},$$
$$(\pm\ddot{u} - \dot{v}) - \sum \frac{\partial F}{\partial r_{ij}}(\pm\dot{u}_{ij} - v_{ij}) \leq A + \frac{K_1\lambda\beta(\beta-2)}{8} \leq 0.$$

The comparison principle of parabolic type ([Fried83]) states that

Lemma 5.7. *Let Ω be a bounded domain of $\mathbb{R}^N$ and $T > 0$. Let $u, v \in C^{2;1}(\Omega \times (0,T]) \cap C(\bar{\Omega} \times [0,T])$. Assume that*

$$\frac{\partial(u-v)}{\partial t} - \sum a_{ij}\frac{\partial^2(u-v)}{\partial x_i \partial x_j} - b.(u-v) \leq 0,$$

where $a_{ij}, b \in C(\Omega \times (0,T))$, $(a_{ij}(x,t))$ are positive definite symmetric matrices and $b(z,t) < 0$. Then $(u-v) \leq max(0, \sup\limits_{\partial_P(\Omega\times(0,T))}(u-v))$.

Applying the comparison principle, we have

$$(\dot{u} - \dot{u}(0,t_0))|_{(\Omega\cap B_R)\times(\epsilon,t_0)} \leq v|_{(\Omega\cap B_R)\times(\epsilon,t_0)}.$$

Hence there exists $K > 0$ such that

$$|\dot{u}(x,t) - \dot{u}(0,t_0)| \leq K(|x| + |t-t_0|^{1/2})^{\beta/2},$$

where $x \in \Omega \times B_R$ and $\epsilon < t \leq t_0$.
Note that R is independent of x_0 and K is independent of t_0. Then there exists $C_{\epsilon,T}$ such that

$$\frac{|\dot{u}(x,t) - \dot{u}(x_0,t_0)|}{(|x-x_0| + |t-t_0|^{1/2})^{\beta/2}} \leq C_\epsilon, \quad \forall x \in \Omega, x_0 \in \partial\Omega; \forall t, t_0 \in (2\epsilon, T).$$

□

Lemma 5.8. *There exists $C_{\epsilon,T} > 0$ depending on $\lambda, \Lambda, \Omega, C, \epsilon, T$ and upper bound of $\|\varphi\|_{C^4} + \|F\|_{C^1} + \|f\|_{C^2}$ such that*

$$u_{\xi\xi}(x,t) - u_{\xi\xi}(x_0,t_0) \leq C_{\epsilon,T}(|x-x_0| + |t-t_0|^{1/2})^{\beta/2}$$

for any $\xi \in \mathbb{R}^N, |\xi| = 1, x \in \Omega, x_0 \in \partial\Omega, \epsilon < t, t_0 < T$. Where $0 < \beta < 1$ is the constant in Lemma 5.3.

Proof. By the equation (29), we have

$$\dot{u}_{\xi\xi} - \sum \frac{\partial F}{\partial r_{ij}}(u_{\xi\xi})_{ij} - f_u.u_{\xi\xi} = \sum \frac{\partial^2 F}{\partial r_{ij}\partial r_{kl}}(u_\xi)_{ij}(u_\xi)_{kl} + O(1) \leq O(1)$$

By Lemma 5.3, we also obtain

$$(u_{\xi\xi}(x,t) - u_{\xi\xi}(x_0,t_0))|_{\partial_P(\Omega\times(\epsilon,T))} = O(|x-x_0| + |t-t_0|^{1/2})^{\beta/2})$$

Then, the proof of Lemma 5.8 is similar to the proof of Lemma 5.6 with the same type of fuction v. □

Lemma 5.9. *There exists $C_{\epsilon,T} > 0$ depending on $\lambda, \Lambda, \Omega, C, \epsilon, T$ and upper bound of $\|\varphi\|_{C^4} + \|F\|_{C^1} + \|f\|_{C^2}$ such that*

$$\|D^2u(x,t) - D^2u(x_0,t_0)\| \leq C_{\epsilon,T}(|x-x_0| + |t-t_0|^{1/2})^{\beta/2}$$

for any $x \in \Omega, x_0 \in \partial\Omega, \epsilon < t, t_0 < T$, where $0 < \beta < 1$ is the constant in Lemma 5.3.

Proof. Let $\lambda_1, ..., \lambda_N$ be eigenvalues of $D^2u(x,t) - D^2u(x_0,t_0)$. We have

$$\|D^2u(x,t) - D^2u(x_0,t_0)\| \leq \sum |\lambda_i|.$$

Moreover,

$$\begin{aligned} \dot{u}(x,t) - f(t,x,u(x,t)) &= F(D^2u(x,t)) \\ &\leq F(D^2u(x_0,t_0)) + \Lambda \sum_{\lambda_i > 0} \lambda_i + \lambda \sum_{\lambda_i < 0} \lambda_i \\ &= \dot{u}(x_0,t_0) - f(t_0,x_0,u(x_0,t_0)) + \Lambda \sum_{\lambda_i > 0} \lambda_i + \lambda \sum_{\lambda_i < 0} \lambda_i. \end{aligned}$$

Hence, by Lemma 5.6 , we have

$$\Lambda \sum_{\lambda_i > 0} |\lambda_i| \geq \lambda \sum_{\lambda_i < 0} |\lambda_i| - A(|x-x_0| + |t-t_0|^{1/2})^{\beta/2},$$

where $A > 0$ is a universal constant.
Then

$$\|D^2u(x,t) - D^2u(x_0,t_0)\| \leq \frac{\Lambda + \lambda}{\lambda} \sum_{\lambda_i > 0} |\lambda_i| + \frac{A}{\lambda}(|x-x_0| + |t-t_0|^{1/2})^{\beta/2}.$$

Note that

$$\sum_{\lambda_i > 0} |\lambda_i| \leq N \max\{0, \lambda_1, ...\lambda_N\} \leq N \max\{\sup_{|\xi|=1}(u_{\xi\xi}(x,t) - u_{\xi\xi}(x_0,t_0)), 0\}.$$

By Lemma 5.8, there exists $C_{\epsilon,T} > 0$ depending on $\lambda, \Lambda, \Omega, C, \epsilon, T$ and upper bound of $\|\varphi\|_{C^4} + \|F\|_{C^1} + \|f\|_{C^2}$ such that

$$\|D^2u(x,t) - D^2u(x_0,t_0)\| \leq C_{\epsilon,T}(|x-x_0| + |t-t_0|^{1/2})^{\beta/2}$$

for any $x \in \Omega, x_0 \in \partial\Omega, \epsilon < t, t_0 < T$.

□

Proof of Theorem 5.1. We need to show that

$$\|D^2u(x,t_1) - D^2u(y,t_2)\| \leq C(|x-y| + |t_1-t_2|^{1/2})^{\gamma}, \tag{40}$$

where $x, y \in \Omega$, $2\epsilon < t_1, t_2 < T - \epsilon$. C and γ are universal constants.
We can assume that $d_x := d(x, \partial\Omega) \geq d_y := d(y, \partial\Omega)$.
If $|x-y|^2 + |t_1 - t_2| \leq \min\{\frac{d_x^2}{4}, \frac{\epsilon}{2}\})$, we denote

$$v(\xi, t) = \frac{1}{a^2}\left(u(x + a.\xi, t_1 + a^2t) - u(x,t_1) - a\sum u_k(x,t_1)\xi_k\right),$$

where $a = \min\{d_x, \epsilon^{1/2}\}$. Then $v \in C^\infty(\mathbb{B} \times (-1,1))$ satisfies

$$\dot{v} = F(D^2v) + f(t_1 + a^2t, x_1 + a\xi, u(x_1 + a\xi, t_1 + a^2t)) = F(D^2v) + \tilde{f}(t,\xi).$$

It follows from the interior estimate (see the theorem 14.7 and the lemma 14.8 of [Lieb96]) that

$$\|v\|_{C^{2,\gamma}(\mathbb{B}_{1/2}\times(-1/2,1/2))} \leq A(\|v\|_{C^2(\mathbb{B}\times(-1,1))}+1),$$

where A is universal, $\gamma=\min\{\alpha,\beta/2\}$, β is the constant in Lemma 5.3 and α is the constant in Theorem 14.7 of [Lieb96].
Moreover

$$\begin{aligned}|v(\xi,t)| &\leq \frac{|u(x+a\xi,t_1+a^2t)-u(x+a\xi,t_1)|}{a^2}\\ &+\frac{|u(x+a\xi,t_1)-u(x,t_1)-a\sum u_k(x,t_1)\xi_k|}{a^2}\\ &\leq \sup|\dot{u}|+\sup\|D^2u\|,\end{aligned}$$

$$|\dot{v}(\xi,t)|=|\dot{u}(x+a\xi,t_1+a^2t)|\leq\sup|\dot{u}|,$$

$$\|D^2v(\xi,t)\|=\|D^2u(x+a\xi,t_1+a^2t)\|\leq\sup\|D^2u\|.$$

Hence

$$\|v\|_{C^{2,\gamma}(\mathbb{B}_{1/2}\times(-1/2,1/2))}\leq B,$$

where B is universal.
Then

$$\|D^2u(x,t_1)-D^2u(y,t_2)\|\leq B(|x-y|+|t_1-t_2|^{1/2})^{\gamma}.$$

If $|x-y|^2+|t_1-t_2|\geq\frac{\epsilon}{2}$, then

$$\|D^2u(x,t_1)-D^2u(y,t_2)\|\leq 2(\frac{\epsilon}{2})^{-\gamma/2}(\sup\|D^2u\|)(|x-y|+|t_1-t_2|^{1/2})^{\gamma}.$$

If $\frac{\epsilon}{2}>|x-y|^2+|t_1-t_2|\geq\frac{d_x^2}{4}$, it follows from Lemma 5.9 that

$$\begin{aligned}\|D^2u(x,t_1)-D^2u(y,t_2)\| &\leq \|D^2u(x,t_1)-D^2u(x_0,t_1)\|+\|D^2u(x_0,t_1)-D^2u(y,t_2)\|\\ &\leq C_{\epsilon,T}(|x-x_0|^{\beta/2}+(|x_0-y|+|t_1-t_2|^{1/2})^{\beta/2})\\ &\leq C(|x-y|+|t_1-t_2|^{1/2})^{\beta/2}\\ &\leq C(|x-y|+|t_1-t_2|^{1/2})^{\gamma}\end{aligned},$$

where $C_{\epsilon,T}$ is the constant in Lemma 5.9, $x_0\in\partial\Omega$ satisfies $d_x=|x-x_0|$ and C is universal. ☐

5.3. **Higher regularity.**

Let $g\in C^{k+1,\alpha}(\bar{\Omega}\times[0,T))$, where $k\geq 0, 0<\alpha<1$. Let F be a function defined on $Mat(N\times N,\mathbb{R})\times\bar{\Omega}\times[0,T)$ such that $F(.,x,t)$ is concave and satisfies (28). Assume that $F\in C^{k+2;k+1,\alpha}(Mat(N\times N,\mathbb{R})\times\bar{\Omega}\times[0,T))$, i.e., the derivaties $D_r^iD_x^jD_t^lF$ are continuous for all $|i|\leq k+2, |j|+2l\leq k+1$ and satisfy

$$\|F\|_{C^{k+2;k+1,\alpha}(Mat(N\times N,\mathbb{R})\times\bar{\Omega}\times[0,T))}=\sum_{|i|\leq k+2}\sup_{r\in Mat(N\times N,\mathbb{R})}|D_r^iF(r,.)|_{C^{k+1,\alpha}(\bar{\Omega}\times[0,T))}<\infty.$$

We consider the $C^{k+3,\alpha}$ regularity of a solution u of the equation

$$\dot{u}=F(D^2u,x,t)+g(x,t). \tag{41}$$

The following boundary estimates hold:

Proposition 5.10. *Let $x_0 \in \partial\Omega$, $k \geq 0$, $r > 0$ and $u \in C^\infty((\Omega \cap B_r(x_0)) \times (0,T)) \cap C^{k+2,\alpha}((\Omega \cap B_r(x_0)) \times (0,T))$ be a solution of*

$$\begin{cases} \dot{u} = F(D^2u, x, t) + g(x,t) \ \ on \ \ (\Omega \cap B_r(x_0)) \times (0,T), \\ u = \varphi \ \ on \ \ (\partial\Omega \cap B_r(x_0)) \times (0,T), \end{cases} \tag{42}$$

where $\varphi \in C^{k+3,\alpha}(\bar{\Omega} \times (0,T)$. Then there exists $r' \in (0,r)$ depending on r, Ω such that $u \in C^{3+k,\alpha}((\Omega \cap B_{r'}(x_0)) \times (\epsilon, T'))$ for any $0 < \epsilon < T' < T$. Moreover

$$\|u\|_{C^{k+3,\alpha}((\Omega \cap B_{r'}(x_0)) \times (\epsilon,T'))} \leq K,$$

where $K > 0$ depends on $\lambda, \Lambda, \alpha, \Omega, \epsilon, T', T, r, r', \|u\|_{C^{k+2,\alpha}}, \|F\|_{C^{k+2;k+1,\alpha}}, \|g\|_{C^{k+1,\alpha}}$, $\|\varphi\|_{C^{k+3,\alpha}}$.

This regularity is proved, for example, in [Lieb96] (or [GT83] , [CC95] for the elliptic version). For the reader's convenience, we recall the arguments here.

Proof. Using a smooth diffeomorphism (as proof of Lemma 5.3), we can replace $\Omega \cap B_r(x_0)$ by B_4^+ and replace $\partial\Omega \cap B_r(x_0)$ by Γ_4. We need to show that $u \in C^{k+3,\alpha}(B_1^+ \times (\epsilon, T'))$.
Let $h > 0$ be small and e_l be the l^{th} vector of the standard basis of R^N, $l < N$. We define

$$\begin{aligned} a_{ij}^h(x,t) \ &= \int_0^1 \frac{\partial F}{\partial r_{ij}}(sD^2u(x+he_l,t) + (1-s)D^2u(x,t), x+she_l, t)ds, \\ g^h(x,t) \ &= \frac{g(x+he_l,t) - g(x,t)}{h}, \\ G^h(x,t) \ &= \int_0^1 F_l(sD^2u(x+he_l,t) + (1-s)D^2u(x,t), x+she_l, t)ds, \\ \varphi^h(x,t) \ &= \frac{\varphi(x+he_l,t) - \varphi(x,t)}{h}, \\ v^h(x,t) \ &= \frac{u(x+he_l,t) - u(x,t)}{h}. \end{aligned}$$

For the convenience, we denote $Q_a = B_a^+ \times (0,T)$ for any $a > 0$. Then

$$\|a_{ij}^h\|_{C^{k,\alpha}(Q_2)} + \|g^h\|_{C^{k,\alpha}(Q_2)} + \|G^h\|_{C^{k,\alpha}(Q_2)} + \|v^h\|_{C^{k+1,\alpha}(Q_2)} + \|\varphi^h\|_{C^{k+2,\alpha}(Q_2)} < A,$$

where $A > 0$ depends only on $\|u\|_{C^{k+2,\alpha}(Q_4)}, \|F\|_{C^{k+2;k+1,\alpha}(Q_4)}, \|g\|_{C^{k+1,\alpha}(Q_4)}, \|\varphi\|_{C^{k+3,\alpha}(Q_4)}$. Moreover,

$$\begin{cases} \dot{v}^h = \sum a_{ij}^h v_{ij}^h + g^h + G^h \ \text{ on } Q_2, \\ v^h = \varphi^h \ \text{ on } \Gamma_2 \times (0,T). \end{cases} \tag{43}$$

If $k = 0$, using a cutoff function and applying Schauder's global estimates ([Fried83],page 65), we have

$$\|v^h\|_{C^{k+2,\alpha}(B_1^+ \times (\epsilon,T'))} \leq C, \tag{44}$$

where $C > 0$ depends on A and ϵ, T'.

If $k > 0$ and Proposition 5.10 is verified for $k-1$, then applying the case $k-1$, we also obtain (44).

It follows that $u_l \in C^{k+2,\alpha}(B_1^+ \times (\epsilon, T'))$ with $\|u_l\|_{C^{k+2,\alpha}(B_1^+ \times (\epsilon,T'))} \leq C$.

By the same method, we can also show that $\|\dot{u}\|_{C^{k+1,\alpha}(B_1^+\times(\epsilon,T'))} \leq C$. It remains to prove $\|u_{NNN}\|_{C^{k,\alpha}(B_1^+\times(\epsilon,T'))} \leq C$. On $B_1^+ \times (\epsilon, T')$, we have

$$\dot{u}_N = \sum(\frac{\partial F}{\partial r_{ij}}(D^2u, x, t))u_{ijN} + F_N(D^2u, x, t) + g_N(x, t).$$

Then

$$u_{NNN} = \frac{1}{\partial F/\partial r_{NN}}\left(\dot{u}_N - \sum_{(i,j)\neq(N,N)}\frac{\partial F}{\partial r_{ij}}u_{ijN} - g_N\right).$$

Note that $\frac{\partial F}{\partial r_{NN}} \geq \lambda > 0$. Hence, $u_{NNN} \in C^{k,\alpha}(B_1^+ \times (\epsilon, T'))$ and $\|u_{NNN}\|_{C^{k,\alpha}(B_1^+\times(\epsilon,T'))}$ is bounded by a universal constant. □

Using the method of the proof above, we also obtain the interior estimates

Proposition 5.11. *Let $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)$. Let $u \in C^{k+2,\alpha}(B_r(x_0)\times(0,T))$ be a solution of*

$$\dot{u} = F(D^2u, x, t) + g(x, t) \ \ \text{on} \ \ B_r(x_0). \tag{45}$$

Then $u \in C^{k+3,\alpha}(B_{r/2}(x_0) \times (\epsilon, T'))$ for any $0 < \epsilon < T' < T$. Moreover

$$\|u\|_{C^{k+3,\alpha}(B_{r/2}(x_0)\times(\epsilon,T'))} \leq C,$$

where $C > 0$ depends on $\lambda, \Lambda, \alpha, \epsilon, T', T, r, \|u\|_{C^{k+2,\alpha}}, \|F\|_{C^{k+2;k+1,\alpha}}, \|g\|_{C^{k+1,\alpha}}$.

Combining Proposition 5.10 and Proposition 5.11, we have the following

Proposition 5.12. *Let F, f, φ be functions defined as 5.2. Assume that $u \in C^{2,\alpha}(\Omega \times (0,T))$ is a solution of*

$$\begin{cases} \dot{u} = F(D^2u) + f(t, x, u) \ \ \text{on} \ \ \Omega \times (0, T), \\ u = \varphi \ \ \text{on} \ \ \partial\Omega \times (0, T). \end{cases} \tag{46}$$

Then $u \in C^\infty(\bar{\Omega} \times (0, T))$.

6. Proof of the main theorem

We recall the main theorem:

Theorem 6.1 (Main theorem)**.** *Let Ω be a bounded smooth strictly pseudoconvex domain of $\mathbb{C}^n$ and $T \in (0, \infty]$. Let u_0 be a bounded plurisubharmonic function defined on a neighbourhood $\tilde{\Omega}$ of $\overline{\Omega}$. Assume that $\varphi \in C^\infty(\bar{\Omega} \times [0, T))$ and $f \in C^\infty([0, T) \times \bar{\Omega} \times \mathbb{R})$ satisfying*

(i) $f_u \leq 0$.
(ii) *$\varphi(z, 0) = u_0(z)$ for $z \in \partial\Omega$.*

Then there exists a unique function $u \in C^\infty(\bar{\Omega} \times (0, T))$ such that

$$u(., t) \text{ is a strictly plurisubharmonic function on } \Omega, \ \ \forall t \in (0, T), \tag{47}$$

$$\dot{u} = \log\det(u_{\alpha\bar{\beta}}) + f(t, z, u) \ \text{on} \ \Omega \times (0, T), \tag{48}$$

$$u = \varphi \ \text{on} \ \partial\Omega \times (0, T), \tag{49}$$

$$\lim_{t\to 0} u(z,t) = u_0(z) \ \ \forall z \in \bar{\Omega}. \tag{50}$$

Moreover, $u \in L^\infty(\bar{\Omega} \times [0,T'))$ for any $0 < T' < T$, and $u(.,t)$ also converges to u_0 in capacity when $t \to 0$.
If $u_0 \in C(\tilde{\Omega})$ then $u \in C(\bar{\Omega} \times [0,T))$.

Proof. Replacing T by $0 < T' < T$, we can assume that $T < \infty$ and there exists C_φ such that

$$\|\varphi\|_{C^4(\Omega\times(0,T))} \leq C_\varphi. \tag{51}$$

We can also assume that $\|f\|_{C^2([0,T)\times\bar{\Omega}\times[-M,M])} < \infty$ for any $M > 0$.
Existence of a solution.
Using the convolution of $u_0 + \frac{|z|^2}{m}$ with smooth kernels, we can take $u_{0,m} \in C^\infty(\bar{\Omega})$ such that

$$u_{0,m} \searrow u_0,$$
$$dd^c u_{0,m} \geq \frac{1}{m} dd^c |z|^2.$$

Note that $u_0|_{\partial\Omega}$ is continuous. Then

$$\delta_m = \sup_{z\in\partial\Omega} (u_{0,m}(z) - u_0(z)) \overset{m\to\infty}{\longrightarrow} 0. \tag{52}$$

We define $g_m \in C^\infty(\bar{\Omega})$ and $\varphi_m \in C^\infty(\bar{\Omega} \times [0,T))$ by

$$g_m = -\log\det(u_{0,m})_{\alpha\bar{\beta}} + f(0,z,u_{0,m}),$$
$$\varphi_m = \zeta(\frac{t}{\epsilon_m})(tg_m + u_{0,m}) + (1 - \zeta(\frac{t}{\epsilon_m}))\varphi,$$

where ζ is a smooth funtion on $\mathbb{R}$ such that ζ is decreasing, $\zeta|_{(-\infty,1]} = 1$ and $\zeta|_{[2,\infty)} = 0$. $\epsilon_m > 0$ are chosen such that the sequences $\{\epsilon_m\}$, $\{\epsilon_m \sup |g_m|\}$ are decreasing to 0 and $\zeta(\frac{t}{\epsilon_m})(u_{0,m}(z) - \varphi(z,t)) \geq 0$ for any m.
Then φ_m converges pointwise to φ on $\partial\Omega \times [0,T)$ and for any $0 < \epsilon < T$, there exists $m_\epsilon > 0$ such that $\varphi_m|_{\bar{\Omega}\times(\epsilon,T)} = \varphi|_{\bar{\Omega}\times(\epsilon,T)}$, $\forall m > m_\epsilon$.
Moveover,

$$\varphi_m(z,0) = u_{0,m}(z) \ \ ,$$
$$\dot{\varphi}_m = \log\det(u_{0,m})_{\alpha\bar{\beta}} + f(t,z,u_{0,m}),$$

where $(z,t) \in \partial\Omega \times \{0\}$.
By the theorem of Hou-Li, there exists $u_m \in C^\infty(\Omega\times(0,T)) \cap C^{2;1}(\bar{\Omega}\times[0,T))$ satisfying

$$\begin{cases} \dot{u}_m = \log\det(u_m)_{\alpha\bar{\beta}} + f(t,z,u_m) & \text{on } \Omega\times(0,T), \\ u_m = \varphi_m & \text{on } \partial\Omega\times[0,T), \\ u_m = u_{0,m} & \text{on } \bar{\Omega}\times\{0\}. \end{cases} \tag{53}$$

Applying Corollary 2.5 for u_1 and u_m, we see that the functions u_m are uniformly bounded by a constant $C_u > 0$. Then we can assume that $\|f\|_{C^2((0,T)\times\Omega\times\mathbb{R})} \leq C_f$. Applying Theorem 1.1 on $\Omega \times (\frac{\epsilon}{2}, T)$, we obtain

$$\|u_m\|_{C^2(\Omega\times(\epsilon,T))} \leq C,$$

where $C = C(\epsilon, T, \Omega, C_f, C_\varphi, C_u)$, m is large enough.
It follows from the $C^{2,\alpha}$ estimates in Section 5 that for any $0 < \epsilon < T' < T$, there exist $M = M(\epsilon, T', C, \Omega, C_\varphi, C_f)$ and $0 < \gamma < 1$ such that

$$\|u_m\|_{C^{2,\gamma}(\bar{\Omega}\times(\epsilon,T))} \leq M.$$

By Ascoli's theorem, there exists $u \in C^{2,\gamma/2}(\bar{\Omega} \times (0,T))$ such that

$$u_{m_k} \overset{C^{2,\gamma/2}(\bar{\Omega}\times(\epsilon,T))}{\longrightarrow} u. \tag{54}$$

Thus u satisfies (47), (48) and (49). By Proposition 5.12 we have $u \in C^\infty(\bar{\Omega} \times (0,T))$. Clearly, u is bounded. We need to show the convergence of $u(.,t)$ when $t \to 0$.
Step 1: $\liminf\limits_{t\to 0} u(z,t) \geq u_0(z)$.
By (54), there exists a subsequence of (u_m), also denoted by (u_m), which converges pointwise to u on $\bar{\Omega} \times (0,T)$.
For any $a > 0$, there exists $A > 0$ such that $\forall m > 0, v_m = u_{0,m} + a\rho - At$ satisfies

$$\begin{cases} \dot{v}_m \leq \log\det(v_m)_{\alpha\bar{\beta}} + f(t,z,v_m), \\ v_m|_{\partial_P(\Omega\times(0,T))} \leq u_m|_{\partial_P(\Omega\times(0,T))} + \epsilon_m \sup|g_m| + \delta_m, \end{cases} \tag{55}$$

where $\rho \in C^\infty(\bar{\Omega})$ is a non-positive strictly plurisubharmonic function on Ω.
It follows from Corollary 2.5 that

$$v_m \leq u_m + \epsilon_m \sup|g_m| + \delta_m.$$

Hence

$$u(z,t) \geq \lim_{m\to\infty}(v_m(z,t) - \epsilon_m \sup|g_m| - \delta_m) = u_0(z) + a\rho(z) - At. \tag{56}$$

Then we have

$$\liminf_{t\to 0} u(z,t) \geq u_0(z) + a\rho(z).$$

When $a \to 0$, we obtain

$$\liminf_{t\to 0} u(z,t) \geq u_0(z). \tag{57}$$

Step 2: $\limsup\limits_{t\to 0} u(z,t) \leq u_0(z)$.
Let $\epsilon > 0$. Assume that $m_0 > 0$ satisfies $\epsilon_{m_0} \sup|g_{m_0}| \leq \epsilon$.
For any $m > k > m_0$, we have

$$\begin{array}{ll} u_{0,m} - u_{0,k} & \leq 0; \\ \varphi_m - \varphi_k & = \zeta(\frac{t}{\epsilon_m})(u_{0,m} - \varphi) - \zeta(\frac{t}{\epsilon_k})(u_{0,k} - \varphi) \\ & +tg_m\zeta(\frac{t}{\epsilon_m}) - tg_k\zeta(\frac{t}{\epsilon_k}) \\ & \leq \zeta(\frac{t}{\epsilon_k})(u_{0,m} - \varphi) - \zeta(\frac{t}{\epsilon_k})(u_{0,k} - \varphi) + 2\epsilon \\ & \leq \zeta(\frac{t}{\epsilon_k})(u_{0,m} - u_{0,k}) + 2\epsilon \\ & \leq 2\epsilon. \end{array}$$

It follows Corollary 2.5 that

$$u_m \leq u_k + 2\epsilon.$$

Hence

$$u(z,t) = \lim_{m\to\infty} u_m(z,t) \leq u_k(z,t) + 2\epsilon. \tag{58}$$

Then we have

$$\limsup_{t\to 0} u(z,t) \leq u_{0,k}(z) + 2\epsilon.$$

When $k \to \infty$ and $\epsilon \to 0$, we obtain

$$\limsup_{t\to 0} u(z,t) \leq u_0(z). \tag{59}$$

Combining (57) and (59), we obtain (50).

Step 3: Convergence in capacity.

The bounded plurisubharmonic function u_0 is continuous outside sets of arbitrarily small capacity. Then the convergence in capacity is implied by (56), (58) and Hartogs lemma (Lemma 90 of [Ber13]) .

If $u_0 \in C(\tilde{\Omega})$ then $u_{0,m}$ and φ_m converge uniformly, respectively, to u_0 and φ. It follows Corollary 2.5 that u_m converges uniformly to u. So u is continuous on $\bar{\Omega} \times [0,T)$.

Uniqueness of the solution.

Let $u, v \in C^\infty(\bar{\Omega} \times (0,T))$ be functions satisfying (47), (48), (49), (50). Let $\epsilon > 0$. We need to show that $u \leq v + (t+3)\epsilon$.

Step 1. $\exists A > 0, v(z,t) \geq u_0(z) - \epsilon - At$.

For $m > 0$, we denote $v_m(z,t) = v(z, t + \frac{1}{m})$. Then v_m is the solution of

$$\begin{cases} \dot{v}_m = \log\det(v_m)_{\alpha\bar{\beta}} + f(t + \frac{1}{m}, z, v_m) \text{ on } \Omega \times (0, T - \frac{1}{m}), \\ v_m(z,t) = \varphi(z, t + \frac{1}{m}) \text{ on } \partial\Omega \times (0, T - \frac{1}{m}). \end{cases} \tag{60}$$

Let $\rho \in C^\infty(\bar{\Omega})$ be a non-positive strictly plurisubharmonic function on Ω such that $\inf \rho = -1$. Then there exists $A > 0$ depending only on $\epsilon, \rho, \|\varphi\|_{C^1}, \sup f(t, z, \sup\varphi)$ such that

$$\begin{cases} \dot{w}_m \leq \log\det(w_m)_{\alpha\bar{\beta}} + f(t + \frac{1}{m}, z, w_m) \text{ on } \Omega \times (0, T - \frac{1}{m}), \\ w_m(z,t) \leq \varphi(z, t + \frac{1}{m}) \text{ on } \partial\Omega \times (0, T - \frac{1}{m}), \end{cases} \tag{61}$$

where $w_m = v(z, \frac{1}{m}) + \epsilon\rho - At$.

Applying Corollary 2.5, we have $v_m \geq w_m$. When $m \to \infty$, we obtain

$$v(z,t) \geq u_0(z) + \epsilon\rho(z) - At \geq u_0(z) - \epsilon - At.$$

Step 2. $\exists m_0 > 0, \forall m > m_0, \exists k_m > m, v(z, \frac{1}{m}) \geq -3\epsilon + u(z, \frac{1}{k_m})$.

Step 1 implies that v is bounded. Then we can assume that $\|f\|_{C^2([0,T)\times\bar{\Omega}\times\mathbb{R})} < \infty$.

By step 1, we have

$$v(z, \frac{1}{m}) + \epsilon + \frac{A}{m} \geq u_0(z) = \lim_{t\to 0} u(z,t).$$

Applying Hartogs lemma, for any $K \Subset \Omega$ there exists $k_{m,K} > m$ such that

$$u(z, \frac{1}{k_{m,K}}) \leq v(z, \frac{1}{m}) + 2\epsilon + \frac{A}{m} \ \ \forall z \in K. \tag{62}$$

Let $m_0 \geq \frac{1}{\epsilon}\max\{1, A, \|f\|_{C^2}, \|h\|_{C^2}\}$, where $h \in C^\infty(\bar{\Omega} \times [0,T))$ is a spatial harmonic function such that $h|_{\partial\Omega\times(0,T)} = \varphi|_{\partial\Omega\times(0,T)}$.

For any $m > m_0$, let $K = K_m \Subset \Omega$ such that

$$v(z, \frac{1}{m}) + \epsilon \geq h(z, \frac{1}{m}) \ \ \forall z \in \Omega \setminus K.$$

Let $k_m = k_{m,K_m}$. Then

$$v(z,\frac{1}{m}) \geq -2\epsilon + h(z,\frac{1}{k_m}) \geq -2\epsilon + u(z,\frac{1}{k_m}) \ \ \forall z \in \Omega \setminus K. \tag{63}$$

Combining (62) and (63), we obtain

$$v(z,\frac{1}{m}) \geq -3\epsilon + u(z,\frac{1}{k_m}) \ \ \forall z \in \Omega.$$

Step 3. Conclusion.

Let $u_m(z,t) = u(z,t+\frac{1}{k_m}) - \epsilon t$. For $m > m_0$, we have

(64)

$$\begin{cases} \dot{v}_m = \log\det(v_m)_{\alpha\bar{\beta}} + f(t+\frac{1}{m},z,v_m) \geq \log\det(v_m)_{\alpha\bar{\beta}} + f(t+\frac{1}{k_m},z,v_m) - \epsilon, \\ \dot{u}_m \leq \log\det(u_m)_{\alpha\bar{\beta}} + f(t+\frac{1}{k_m},z,u_m) - \epsilon. \end{cases}$$

Applying Corollary 2.5, we have

$$(u_m - v_m) \leq \sup_{\partial_P(\Omega\times(0,T-\frac{1}{m}))} (u_m - v_m) \leq 3\epsilon$$

When $m \to \infty$, we have

$$u(z,t) - v(z,t) - \epsilon t = \lim_{m\to\infty} (u_m(z,t) - v_m(z,t)) \leq 3\epsilon.$$

When $\epsilon \to 0$, we obtain

$$u(z,t) \leq v(z,t).$$

Since the roles of u and v are symmetric, $v(z,t) \leq u(z,t)$. Then $u = v$. $\square$

7. Further directions

In this section, we discuss further questions in the same general directions as our result. On compact Kähler manifolds, the corresponding problem was solved in the case where $f = 0$ and u_0 has zero Lelong numbers. In that case, there exists a solution u satisfying $u(.,t) \to u_0$ in L^1 (see [GZ13]), and the solution is unique (see [DL14]). It is natural to ask whether the same result holds for a domain in $\mathbb{C}^n$. Let us state our conjecture

Conjecture 7.1. *If we replace the condition "$u_0 \in L^\infty(\tilde{\Omega})$" in Theorem 6.1 by the condition "u_0 has zero Lelong numbers" then there exists a unique function $u \in C^\infty(\bar{\Omega}\times(0,T))$ satisfying* (47), (48), (49) *such that $u(.,t) \to u_0$ in $L^1(\Omega)$.*

The case where u_0 has positive Lelong numbers is another problem. It was also considered and solved in the case compact Kähler manifold by [GZ13] and [DL14]. It is the motivation of the second direction: the case of domain in $\mathbb{C}^n$ and u_0 has positive Lelong numbers.

There is another question: What is the behavior when we replace the condition "$u_0 \in PSH(\tilde{\Omega})$" in Theorem 6.1 by the condition "$u_0 \in PSH(\Omega)$"? In order to prove Theorem 6.1, we construct plurisubharmonic functions $u_{0,m}$ which converge to u_0. This step is easy if we suppose that $u_0 \in PSH(\tilde{\Omega})$. If we only suppose that "$u_0 \in PSH(\Omega)$ and $\lim_{z\to z_0\in\partial\Omega} u_0(z) = \varphi(z_0)$", maybe this step is still realizable but more difficult. We give a provisional result in this direction.

Proposition 7.2. *Let Ω be a bounded smooth strictly pseudoconvex domain of $\mathbb{C}^n$ and $T \in (0,\infty]$. Let u_0 be a continuous plurisubharmonic function on Ω such that u_0 is smooth on $\bar{\Omega}\backslash\mathcal{K}$, where $\mathcal{K} \Subset \Omega$. Assume that φ, f are functions satisfying the conditions of Theorem 6.1. Then there exists a unique function $u \in C^\infty(\bar{\Omega}\times(0,T))\cap C(\bar{\Omega}\times[0,T))$ satisfying* (47), (48), (49) *and $u(.,0) = u_0$.*

Proof sketch. Let ρ, ζ be the functions defined in the proof of Theorem 6.1. Let ψ be a smooth function in Ω and ϕ be a smooth function on $\mathbb{R}$ satisfying

- $0 \leq \psi \leq 1$, $\psi|_{U_1} = 1, \psi|_{\Omega\backslash U_2} = 0$, where $\mathcal{K} \Subset U_1 \Subset U_2 \Subset \Omega$.
- ϕ is convex and increasing, $\phi|_{(-\infty,-3)} = -2$, $\phi|_{(-1,\infty)} = Id$.

Using convolutions of $u_0 + \frac{\rho}{m}$, we can find $\tilde{u}_{0,m} \in C^\infty(U_2)$ such that $\tilde{u}_{0,m}$ and $\psi\tilde{u}_{0,m} + (1-\psi)(u_0 + \frac{\rho}{m})$ are strictly plurisubharmonic functions.
We define $u_{0,m} \in C^\infty(\bar{\Omega})$, $g_m \in C^\infty(\bar{\Omega}\setminus\mathcal{K})$, $\varphi_m \in C^\infty(\bar{\Omega}\times[0,T))$ by

$$u_{0,m} = \psi\tilde{u}_{0,m} + (1-\psi)(u_0 + \frac{\rho}{m}) + \frac{1}{m}\phi\circ(m\rho),$$

$$g_m = -\dot{\varphi}|_{t=0} + \log\det(u_0 + \frac{m+1}{m}\rho)_{\alpha\bar{\beta}} + f(t, z, u_0 + \frac{m+1}{m}\rho),$$

$$\varphi_m = (1-\psi)(t\zeta(mt)g_m + u_0 + \frac{m+1}{m}\rho + \int_0^t \dot{\varphi}).$$

Repeating the techniques in the proof of Theorem 6.1, we show that there exists a unique function $u \in C^\infty(\bar{\Omega}\times(0,T))\cap C(\bar{\Omega}\times[0,T))$ satisfying (47), (48), (49) such that $u|_{t=0} = u_0$.

□

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E-mail address: hoangson.do.vn@gmail.com